



# $n$ -Weakly hypercyclic and $n$ -weakly supercyclic operators

Nathan S. Feldman

*Dept. of Mathematics, Washington and Lee University, Lexington, VA 24450, United States*

Received 23 September 2011; accepted 11 July 2012

Available online 20 August 2012

Communicated by K. Ball

## Abstract

If  $X$  is a locally convex topological vector space over a scalar field  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$  and if  $E$  is a subset of  $X$ , then we define  $E$  to be  $n$ -weakly dense in  $X$  if for every onto continuous linear operator  $F : X \rightarrow \mathbb{F}^n$  we have that  $F(E)$  is dense in  $\mathbb{F}^n$ . If  $X$  is a Hilbert space, this is equivalent to requiring that  $E$  have a dense orthogonal projection onto every subspace of dimension  $n$ . We then consider continuous linear operators on  $X$  that have orbits or scaled orbits that are  $n$ -weakly dense in  $X$ . We show that on a separable Hilbert space there are non-trivial examples of such operators and establish many of their basic properties. A fundamental tool is Ball's solution of the complex plank problem which implies that certain sets are 1-weakly closed.

© 2012 Elsevier Inc. All rights reserved.

**Keywords:** Hypercyclic; Supercyclic; Weakly hypercyclic; Weakly supercyclic

## 1. Introduction

Suppose that  $T$  is a continuous linear operator on a topological vector space  $X$  over a field  $\mathbb{F}$  where  $\mathbb{F}$  will denote either the field of real numbers  $\mathbb{R}$  or the field of complex numbers  $\mathbb{C}$ . For an element  $x \in X$ , the orbit of  $x$  under  $T$  is  $\text{Orb}(x, T) = \{T^n x\}_{n=0}^{\infty} = \{x, Tx, T^2x, \dots\}$ . The operator  $T$  is called *hypercyclic* if it has a dense orbit, that is, if there is a vector  $x \in X$  such that  $\text{Orb}(x, T)$  is dense in  $X$ . Likewise  $T$  is called *supercyclic* if it has a scaled orbit that is dense in  $X$ ; meaning, there exists a vector  $x \in X$  such that  $\mathbb{F} \cdot \text{Orb}(x, T)$  is dense in  $X$ .

---

*E-mail address:* [feldmanN@wlu.edu](mailto:feldmanN@wlu.edu).

In 1969 Rolewicz [21] gave the first example of a hypercyclic operator on a Banach space; namely twice the backward shift on  $\ell^2(\mathbb{N})$ . Later in 1974, Hilden and Wallen [18] introduced supercyclic operators and proved that every backward unilateral weighted shift is supercyclic. Since that time there has been a large amount of research devoted to hypercyclic and supercyclic operators. See, for instance, the two recent books on hypercyclicity [5] and [17], as well as their references.

If  $E$  is a subset of  $X$ , then we say that  $E$  is  $n$ -weakly dense in  $X$  if  $F(E)$  is dense in  $\mathbb{F}^n$  for every onto continuous linear operator  $F : X \rightarrow \mathbb{F}^n$ . An important example of an  $n$ -weakly dense set in  $\ell^2(\mathbb{N})$  is the set of all vectors with at most  $n$  non-zero coordinates (see Corollary 2.10 and Proposition 6.8).

If  $n$  is a positive integer, then we say that the operator  $T$  is  $n$ -weakly hypercyclic if there is a vector  $x \in X$  such that  $\text{Orb}(x, T)$  is  $n$ -weakly dense in  $X$ . Similarly, we say that  $T$  is  $n$ -weakly supercyclic if there is a vector  $x \in X$  such that  $\mathbb{F} \cdot \text{Orb}(x, T)$  is  $n$ -weakly dense in  $X$ .

In this paper we will show that there are non-trivial examples of  $n$ -weakly hypercyclic and  $n$ -weakly supercyclic operators on Hilbert space and establish some basic properties of these operators.

Two important criteria for constructing  $n$ -weakly hypercyclic/supercyclic operators are in Theorems 6.3 and 10.5. These allow us to construct vectors the closure of whose orbits contains a prescribed set. We use these to provide a general method (see Theorem 6.10) for constructing  $n$ -weakly hypercyclic operators by simply taking direct sums of hypercyclic operators that satisfy a strong form of the hypercyclicity criterion. In particular, if  $T_1, T_2, \dots, T_p$  are each hypercyclic operators satisfying the strong hypercyclicity criterion (see Definition 6.6) and if the direct sum of any  $n$  ( $1 \leq n < p$ ) of the operators  $\{T_1, \dots, T_p\}$  is hypercyclic, then the direct sum of all the operators,  $\bigoplus_{k=1}^p T_k$ , is  $n$ -weakly hypercyclic. This allows us to show that for every positive integer  $n$ , there exist Hilbert space operators that are  $n$ -weakly hypercyclic but not  $(n+1)$ -weakly hypercyclic (see Theorem 6.13).

We also give similar conditions for the direct sum of operators to be  $n$ -weakly supercyclic. It is shown that there are Hilbert space operators  $T$  (of the form  $T_1 \oplus I_1$  where  $T_1$  is  $n$ -weakly hypercyclic and  $I_1$  is the identity operator on a one-dimensional space) such that  $T$  is  $n$ -weakly supercyclic but not  $(n+2)$ -weakly supercyclic (see Corollary 7.7). Another example shows that the direct sum of certain bilateral weighted shifts is  $n$ -weakly supercyclic, but not  $(2n+2)$ -weakly supercyclic (see Corollary 10.9). Many other structural properties are established.

We show that a 1-weakly hypercyclic operator must have each component of its spectrum intersecting the unit circle (see Proposition 3.3). An  $n$ -weakly hypercyclic operator  $T$  cannot have a finite set of non-zero vectors with cardinality at most  $n$  whose orbit under  $T^*$  is jointly bounded (see Proposition 3.6). In Section 4 we summarize the results of Feldman [15] describing which matrices are  $n$ -weakly supercyclic. In Theorem 5.8 we present an example which provides a step towards formulating the proper version of Ansari's Theorem for  $n$ -weakly hypercyclic operators. This example also shows how to construct orbits for twice the backward shift that are  $n$ -weakly dense but not dense.

In Proposition 7.1 we establish the simple fact that an operator is 1-weakly supercyclic if and only if it is cyclic. We also show that pure subnormal operators cannot be 2-weakly supercyclic (see Corollary 7.3 and Corollary 7.15) and that a 2-weakly supercyclic bilateral weighted shift must be supercyclic (see Corollary 7.5).

In Theorem 7.8 we present the  $n$ -weak angle criterion which implies that certain operators cannot be  $(2n)$ -weakly supercyclic. This generalizes the weak angle criterion (Shkarin [24]) which says certain operators cannot be weakly supercyclic.

In Theorem 8.1 we show that an absolutely continuous unitary operator cannot be 2-weakly supercyclic. However, it was shown in Bayart and Matheron [4] that certain singular unitary operators are weakly supercyclic. In Theorem 9.6 it is shown how to construct sequences of operators that are  $n$ -weakly universal. In Theorem 10.3 and Theorem 10.5 we present criteria for showing that certain operators are  $n$ -weakly supercyclic. These are then applied to show that certain direct sums of bilateral weighted shifts are  $n$ -weakly supercyclic.

A nice generalization of the well-known fact that the weak closure of a convex set is the same as its norm closure states that the 1-weak closure of a convex set is the same as its norm closure (see Proposition 2.3).

A beautiful and useful theorem due to Ball states that if a sequence  $\{x_n\}_{n=1}^\infty$  of vectors in a Banach space satisfies  $\sum_{n=1}^\infty \frac{1}{\|x_n\|} < \infty$ , then the set  $\{x_n\}_{n=1}^\infty$  is 1-weakly closed in  $X$  (see Theorem 3.1 and Definition 2.1). This deep theorem will be used repeatedly. We close with a list open questions and conjectures.

## 2. Preliminaries

In what follows  $X$  and  $Y$  will generally denote locally convex topological vector spaces over either the field  $\mathbb{R}$  of real numbers or the field  $\mathbb{C}$  of complex numbers. We will let  $\mathbb{F}$  denote either the set of real or complex numbers. The dual space  $X^*$  denotes the set of all continuous linear functionals on  $X$ . Also,  $\mathcal{B}(X)$  will denote the set of all continuous linear operators on  $X$  and  $\mathcal{B}(X, Y)$  will denote the set of all continuous linear operators from  $X$  to  $Y$ .

If  $X$  is a locally convex space and if  $x_0 \in X$ , then a basis for the *weak topology* on  $X$  at the point  $x_0$  is given as follows:

For  $\mathcal{F} = \{f_1, \dots, f_n\} \subseteq X^*$  and  $\varepsilon > 0$ , let

$$N(x_0, \mathcal{F}, \varepsilon) = N(x_0, f_1, \dots, f_n, \varepsilon) = \{x \in X: |f(x) - f(x_0)| < \varepsilon \text{ for all } f \in \mathcal{F}\}.$$

If we let  $F: X \rightarrow \mathbb{F}^n$  be given by  $F(x) = (f_1(x), \dots, f_n(x))$  and let  $\|\cdot\|_\infty$  be the  $\ell^\infty$ -norm on  $\mathbb{F}^n$ , then notice that

$$N(x_0, f_1, \dots, f_n, \varepsilon) = N(x_0, F, \varepsilon) := \{x \in X: \|F(x) - F(x_0)\|_\infty < \varepsilon\}.$$

A set  $E \subseteq X$  is said to be *weakly open* if for every  $x_0 \in E$ , there is a finite set  $\mathcal{F} \subseteq X^*$ , and an  $\varepsilon > 0$ , such that  $N(x_0, \mathcal{F}, \varepsilon) \subseteq E$ .

We now introduce the idea of an  *$n$ -weakly open* set where we limit the size of the set  $\mathcal{F}$ . If  $A$  is a set, then we will use  $|A|$  to denote the cardinality of  $A$ .

**Definition 2.1.** Let  $n$  be a positive integer,  $X$  a locally convex space and  $E \subseteq X$ . Then we have the following definitions.

- (1) The set  $E$  is  *$n$ -weakly open* if for every  $x_0 \in E$  there is an  $\varepsilon > 0$  and a set  $\mathcal{F} \subseteq X^*$  with  $|\mathcal{F}| \leq n$  such that  $N(x_0, \mathcal{F}, \varepsilon) \subseteq E$ .
- (2) The set  $E$  is  *$n$ -weakly closed* if the complement of  $E$  is  $n$ -weakly open.
- (3) The set  $E$  is  *$n$ -weakly dense* in  $X$  if  $E \cap N \neq \emptyset$  for every non-empty  $n$ -weakly open set  $N$  in  $X$ .
- (4) A point  $x_0 \in X$  is in the  *$n$ -weak closure* of a set  $E$  if for every  $n$ -weakly open set  $N$  containing  $x_0$ , we have  $N \cap E \neq \emptyset$ .

Notice that the sets of the form  $N(x_0, \mathcal{F}, \varepsilon)$  are  $n$ -weakly open sets if the cardinality of  $\mathcal{F}$  is at most  $n$ . In fact, we will call them the *basic  $n$ -weakly open sets*; since every  $n$ -weakly open set is a union of basic  $n$ -weakly open sets. Notice that the  $n$ -weakly open sets do not form a topology, since they are not closed under finite intersections and  $n$ -weakly closed sets are not closed under finite unions. The following proposition elaborates more on this point.

**Proposition 2.2** ( *$n$ -Weak “pseudo-topological” properties*). *If  $X$  is a locally convex space, then the following hold:*

- (1) *If  $U$  is an  $m$ -weakly open set in  $X$  and  $V$  is an  $n$ -weakly open set in  $X$ , and  $k = \max\{m, n\}$ , then  $U \cup V$  is a  $k$ -weakly open set in  $X$ .*
- (2) *If  $U$  is an  $m$ -weakly open set in  $X$  and  $V$  is an  $n$ -weakly open set in  $X$ , then  $U \cap V$  is an  $(m + n)$ -weakly open set in  $X$ .*
- (3) *If  $F_1$  is an  $m$ -weakly closed set in  $X$  and  $F_2$  is an  $n$ -weakly closed set in  $X$ , then  $F_1 \cup F_2$  is an  $(m + n)$ -weakly closed set in  $X$ .*
- (4) *If  $F_1$  is an  $m$ -weakly closed set in  $X$  and  $F_2$  is an  $n$ -weakly closed set in  $X$ , and  $k = \max\{m, n\}$ , then  $F_1 \cap F_2$  is a  $k$ -weakly closed set in  $X$ .*

**Proposition 2.3** (*Hahn–Banach separation*). *If  $C$  is a convex set in a locally convex space  $X$ , then the 1-weak closure of  $C$  in  $X$  is equal to the closure of  $C$  in  $X$ .*

**Proof.** The proof of this result is exactly the same as the proof that the weak closure of a convex set is equal to its closure (see Conway [11]). Simply notice that the classic proof only uses one linear functional to establish the result.  $\square$

The following proposition gives various ways to think about and work with  $n$ -weakly dense sets. We leave the proof to the reader.

**Proposition 2.4.** *If  $X$  is a locally convex space over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ,  $E \subseteq X$ , and  $n$  is a positive integer, then the following are equivalent:*

- (1)  *$E$  is  $n$ -weakly dense in  $X$ .*
- (2)  *$F(E)$  is dense in  $\mathbb{F}^n$  for every onto continuous linear map  $F : X \rightarrow \mathbb{F}^n$ .*
- (3)  *$\{(f_1(x), \dots, f_n(x)) : x \in E\}$  is dense in  $\mathbb{F}^n$  for every linearly independent set of functionals  $\{f_1, \dots, f_n\} \subseteq X^*$ .*
- (4)  *$F(E)$  is dense in  $F(X)$  for every continuous linear map  $F : X \rightarrow Y$  with  $\dim(F(X)) \leq n$ .*
- (5)  *$\pi(E)$  is dense in  $X/\mathcal{M}$  for every closed linear subspace  $\mathcal{M}$  of  $X$  satisfying  $\dim(X/\mathcal{M}) \leq n$ , where  $\pi : X \rightarrow X/\mathcal{M}$  is the natural quotient map.*
- (6) *For every subspace  $\mathcal{N} \subseteq X$  of codimension  $n$ , there is a complementary subspace  $\mathcal{M}$  for  $\mathcal{N}$  for which the projection of  $E$  onto  $\mathcal{M}$  along  $\mathcal{N}$  is dense in  $\mathcal{M}$ .*
- (7) *For every subspace  $\mathcal{N} \subseteq X$  of codimension  $n$  and for every complementary subspace  $\mathcal{M}$  for  $\mathcal{N}$ , the projection of  $E$  onto  $\mathcal{M}$  along  $\mathcal{N}$  is dense in  $\mathcal{M}$ .*
- (8) *If  $X$  is a Hilbert space, then the above conditions are also equivalent to: For every subspace  $\mathcal{M}$  of  $X$  with  $\dim(\mathcal{M}) = n$ , the orthogonal projection of  $E$  onto  $\mathcal{M}$  is dense in  $\mathcal{M}$ .*

The following propositions will be used repeatedly; their proofs are elementary and will be left to the reader.

**Proposition 2.5.** Suppose that  $X$  is a locally convex space over the scalar field  $\mathbb{F}$ .

- (1) If  $\{f_1, \dots, f_n\} \subseteq X^*$  and we define a linear operator  $F : X \rightarrow \mathbb{F}^n$  by  $F(x) = (f_1(x), f_2(x), \dots, f_n(x))$ , then  $F$  is onto if and only if the set  $\{f_1, \dots, f_n\}$  of functionals is linearly independent.
- (2) If  $\dim(X) \leq n$ , then a set  $E \subseteq X$  is  $n$ -weakly dense in  $X$  if and only if  $E$  is dense in  $X$ .

**Proposition 2.6.** Suppose that  $X$  is a locally convex space over the scalar field  $\mathbb{F}$ , that  $E \subseteq X$ ,  $x_0 \in X$ , and  $n \geq 1$ . Then  $x_0$  belongs to the  $n$ -weak closure of  $E$  if and only if  $F(x_0)$  is in the closure of  $F(E)$  for every continuous linear map  $F : X \rightarrow \mathbb{F}^n$ .

One should think geometrically of a basic  $n$ -weakly open set  $N$  as an  $\varepsilon$ -neighborhood of an affine subspace  $\mathcal{L}$  of codimension  $n$ . All  $n$ -weakly open sets are unions of such basic  $n$ -weakly open sets. Thus a basic 1-weakly open set in  $\mathbb{R}^2$  is an  $\varepsilon$ -neighborhood of a line, hence a thin strip, meaning the region between two parallel lines. A basic 2-weakly open set in  $\mathbb{R}^2$  is an  $\varepsilon$ -neighborhood of a point. A basic 1-weakly open set in  $\mathbb{R}^3$  is an  $\varepsilon$ -neighborhood of a plane, like a board or a plank, and a basic 2-weakly open set in  $\mathbb{R}^3$  is an  $\varepsilon$ -neighborhood of a line; like a long thin rod.

With these images in mind, it is easy to verify the following elementary examples.

**Example 2.7** (Some simple  $n$ -weakly dense sets in  $\mathbb{R}^n$ ).

- (1) The union of the  $x$  and  $y$  axes form a 1-weakly dense set in  $\mathbb{R}^2$ .
- (2) The union of the  $x$ ,  $y$ , and  $z$  axes in  $\mathbb{R}^3$  form a 1-weakly dense set in  $\mathbb{R}^3$ , that is not 2-weakly dense in  $\mathbb{R}^3$ .
- (3) The union of the three coordinate planes in  $\mathbb{R}^3$  forms a 2-weakly dense set in  $\mathbb{R}^3$  that is not 3-weakly dense (= dense) in  $\mathbb{R}^3$ .
- (4) A union of  $n$  independent one-dimensional subspaces of  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ ) is 1-weakly dense, but not 2-weakly dense, in  $\mathbb{R}^n$  (respectively  $\mathbb{C}^n$ ).

Item (4) above follows from the next proposition.

**Proposition 2.8.** If  $E \subseteq X$  is a subset of a locally convex space  $X$  that is closed under scalar multiplication, then  $E$  is 1-weakly dense in  $X$  if and only if  $E$  has dense linear span in  $X$ .

**Proof.** Notice that by Proposition 2.4,  $E$  is 1-weakly dense in  $X$  if and only if  $f(E)$  is dense in  $\mathbb{F}$  for every non-zero continuous linear functional  $f : X \rightarrow \mathbb{F}$ . On the other hand, by the Hahn–Banach Theorem,  $E$  has dense linear span in  $X$  if and only if  $f(E) \neq \{0\}$  for every non-zero continuous linear functional  $f$  on  $X$ . However, since  $E$  is closed under scalar multiplication we have that  $f(E) \neq \{0\}$  holds if and only if  $f(E)$  is dense. Thus,  $E$  is 1-weakly dense in  $X$  if and only if  $E$  has dense linear span in  $X$ .  $\square$

Notice that the following proposition is a direct generalization of items (1), (2), and (3) in Example 2.7.

**Proposition 2.9.** *If  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ,  $1 \leq n < d$  and*

$$X_n = \{x \in \mathbb{F}^d : x \text{ has at most } n \text{ non-zero coordinates}\}$$

*then  $X_n$  is  $n$ -weakly dense in  $\mathbb{F}^d$  but not  $(n+1)$ -weakly dense in  $\mathbb{F}^d$ .*

**Proof.** Let  $F : \mathbb{F}^d \rightarrow \mathbb{F}^n$  be an onto continuous linear operator, then we must show that  $F(X_n)$  is dense in  $\mathbb{F}^n$ ; in fact we will show that  $F(X_n) = \mathbb{F}^n$ . Since  $F$  is linear and continuous, there is an  $n \times d$  matrix  $M$  such that  $F(x) = Mx$  for all  $x \in \mathbb{F}^d$ . Since  $F$  maps onto  $\mathbb{F}^n$ , then  $M$  must have rank equal to  $n$ , thus  $M$  must have  $n$  linearly independent columns. Suppose that the columns of  $M$  in positions  $k_1, \dots, k_n$  are linearly independent. Since  $X_n$  contains all vectors  $x$  that have arbitrary values in coordinates  $k_1, \dots, k_n$  and are zero elsewhere, it follows easily that  $F(X_n) = M(X_n) = \mathbb{F}^n$ . Thus,  $X_n$  is  $n$ -weakly dense in  $\mathbb{F}^d$ .

To see that  $X_n$  is not  $(n+1)$ -weakly dense in  $\mathbb{F}^d$ , consider the subspace  $\mathcal{M}$  spanned by  $\{e_1, \dots, e_n, e_{n+1}\}$ , which consists of all vectors whose first  $(n+1)$  coordinates are arbitrary and whose other coordinates are all zero. We claim that the orthogonal projection of  $X_n$  onto  $\mathcal{M}$  is not dense in  $\mathcal{M}$ . This is simply because any  $(n+1)$  consecutive coordinates of a vector in  $X_n$  will contain a zero. Thus the orthogonal projection of  $X_n$  onto  $\mathcal{M}$  will consist of vectors each of which will have at least one coordinate equal to zero. Thus the projection of  $X_n$  is contained in the union of  $(n+1)$  hyperplanes in  $\mathcal{M}$  and hence not dense in  $\mathcal{M}$ . Thus,  $X_n$  is not  $(n+1)$ -weakly dense in  $\mathbb{F}^d$ .  $\square$

The Hilbert space of square summable sequences in  $\mathbb{F}$  will be denoted by  $\ell_{\mathbb{F}}^2$  and the vectors  $\{e_k\}$  will denote the standard basis vectors; that is,  $e_k$  is the sequence that is one in the  $k$ th position and zero elsewhere.

**Corollary 2.10.** *If  $n \geq 1$  and*

$$X_n = \{x \in \ell_{\mathbb{F}}^2 : x_k \neq 0 \text{ for at most } n \text{ coordinates}\}$$

*then  $X_n$  is  $n$ -weakly dense in  $\ell_{\mathbb{F}}^2$  but not  $(n+1)$ -weakly dense in  $\ell_{\mathbb{F}}^2$ .*

**Proof.** Suppose that  $F : \ell_{\mathbb{F}}^2 \rightarrow \mathbb{F}^n$  is an onto continuous linear map. We will show that  $F(X_n)$  is dense in  $\mathbb{F}^n$ . Let  $p \in \mathbb{F}^n$  and  $\varepsilon > 0$ . Since  $F$  is onto and since vectors with finite support are dense in  $\ell_{\mathbb{F}}^2$ , choose a vector  $v \in \ell_{\mathbb{F}}^2$  with finite support such that  $F(v)$  is within  $\varepsilon/2$  of  $p$ . Next choose a  $d > n$  large enough so that  $v \in \mathcal{M} := \text{span}\{e_1, \dots, e_d\}$ . By Proposition 2.9, the set  $X_n \cap \mathcal{M}$  is  $n$ -weakly dense in  $\mathcal{M}$ . In particular, by Proposition 2.4 (item (4)),  $F(X_n \cap \mathcal{M})$  is dense in  $F(\mathcal{M})$ . So we can choose a vector  $w \in X_n \cap \mathcal{M}$  so that  $F(w)$  is within  $\varepsilon/2$  of  $F(v)$ . Thus,  $w \in X_n$  and  $F(w)$  is within  $\varepsilon$  of  $p$ . It follows that  $F(X_n)$  is dense in  $\mathbb{F}^n$ . Thus,  $X_n$  is  $n$ -weakly dense in  $\ell_{\mathbb{F}}^2$ .

To see that  $X_n$  is not  $(n+1)$ -weakly dense in  $\ell_{\mathbb{F}}^2$ , simply notice, as in Proposition 2.9, that the orthogonal projection of  $X_n$  onto  $\mathcal{M} = \text{span}\{e_1, e_2, \dots, e_n, e_{n+1}\}$  is not dense in  $\mathcal{M}$ . Thus  $X_n$  is not  $(n+1)$ -weakly dense in  $\ell_{\mathbb{F}}^2$ .  $\square$

### 3. Components of the spectrum and jointly bounded orbits

In this section we will use the deep theorem of K. Ball (see [2] and [3]) to prove that every component of the spectrum of a 1-weakly hypercyclic operator must intersect the unit circle.

We also introduce the idea of a set of vectors having a *jointly bounded* orbit and show that an  $n$ -weakly hypercyclic operator cannot have a set with cardinality at most  $n$  that has a jointly bounded orbit.

The following version of Ball's Theorem is proved in Shkarin [24, p. 62, Proposition 5.2] and in the book by Bayart and Matheron [5, p. 232]. Their statements do not use the term “1-weakly closed”, but their proofs give the desired results.

**Theorem 3.1** (Ball's Theorem). *Let  $S = \{x_n\}_{n=1}^\infty$  be a sequence of non-zero vectors in a Banach space  $X$ .*

- (1) *If  $\sum_{n=0}^\infty \frac{1}{\|x_n\|} < \infty$ , then  $S$  is 1-weakly closed in  $X$ .*
- (2) *If  $X$  is a Hilbert space and  $\sum_{n=0}^\infty \frac{1}{\|x_n\|^2} < \infty$ , then the following hold:*
  - (a) *If  $X$  is a complex Hilbert space, then  $S$  is 1-weakly closed in  $X$ .*
  - (b) *If  $X$  is a real Hilbert space, then  $S$  is 2-weakly closed in  $X$ .*

The following useful proposition will be used repeatedly, its proof is left to the reader.

**Proposition 3.2.**

- (1) *If  $P : X \rightarrow Y$  is a continuous linear operator with dense range and if  $E$  is  $n$ -weakly dense in  $X$ , then  $P(E)$  is  $n$ -weakly dense in  $Y$ .*
- (2) *If  $T$  is  $n$ -weakly hypercyclic (respectively  $n$ -weakly supercyclic) on a Hilbert space, then the compression of  $T$  to every coinvariant subspace is also  $n$ -weakly hypercyclic (respectively  $n$ -weakly supercyclic).*
- (3) *If  $T_1 \in \mathcal{B}(X)$ ,  $T_2 \in \mathcal{B}(Y)$ ,  $T_1$  is  $n$ -weakly hypercyclic (respectively,  $n$ -weakly supercyclic),  $P : X \rightarrow Y$  has dense range and  $PT_1 = T_2P$ , then  $T_2$  is also  $n$ -weakly hypercyclic (respectively  $n$ -weakly supercyclic).*

The following proposition is elementary and uses routine arguments except that item (3) below makes use of Ball's Theorem above.

**Proposition 3.3.** *If  $T$  is 1-weakly hypercyclic, then*

- (1)  *$T^*$  does not have any non-zero bounded orbits.*
- (2)  *$T^*$  does not have any eigenvectors.*
- (3) *Every component of the spectrum of  $T$  must intersect the unit circle.*
- (4) *If  $x$  is an  $n$ -weakly hypercyclic vector for  $T$  and  $p(z)$  is a non-constant polynomial, then  $p(T)x$  is also an  $n$ -weakly hypercyclic vector for  $T$ .*

**Proof.** Items (1) and (2) are elementary and hold for real and complex spaces. For item (3) if  $T$  acts on a complex space, then basic operator theory techniques, including Proposition 3.2, reduce the problem to showing that if either  $\sigma(T) \subseteq \mathbb{D}$  or  $\sigma(T) \subseteq \{z \in \mathbb{C} : |z| > 1\}$ , then  $T$  cannot be 1-weakly hypercyclic. In the first case, all orbits of  $T$  converge to zero and hence  $T$  cannot be 1-weakly hypercyclic. In the second case where the spectrum of  $T$  lies in the exterior of the unit disk, then there is an  $r > 1$  such that  $\sigma(T) \subseteq \{z \in \mathbb{C} : |z| > r\}$ . Then again basic operator theory implies that there is a constant  $c > 0$  such that  $\|T^n x\| \geq c\|x\|r^n$  for all  $n \geq 0$ . In particular it



follows that  $\sum_{n=0}^{\infty} \frac{1}{\|T^n x\|} < \infty$ . Thus by Ball's Theorem (Theorem 3.1) the set  $\{T^n x\}_{n=0}^{\infty}$  is 1-weakly closed in  $X$  and hence cannot be 1-weakly dense in  $X$  (otherwise the orbit would equal  $X$ , but  $X$  is uncountable and the orbit is countable). Thus the result follows. Item (4) is an analogue of Bourdon's Theorem [8] for hypercyclic operators and it follows from (2).  $\square$

**Corollary 3.4.** *A normal operator cannot be 1-weakly hypercyclic.*

**Proof.** If  $N$  is a normal operator that is 1-weakly hypercyclic, then by Proposition 3.3 every component of the spectrum of  $N$  must intersect the unit circle. Hence the spectrum of  $N$  must be contained in the unit circle. However, then all orbits of  $N$  are bounded and thus  $N$  could not be 1-weakly hypercyclic, a contradiction. Thus a normal operator cannot be 1-weakly hypercyclic.  $\square$

Next we will prove a more general version of item (1) above, in Proposition 3.3, that will be useful later on. But first we need a definition.

**Definition 3.5.** If  $T$  is an operator on a Banach space  $X$  and  $\{x_1, x_2, \dots, x_p\} \subseteq X$ , then we will say that the set  $\{x_k\}_{k=1}^p$  has a *jointly bounded orbit* under  $T$  if the following holds:

$$\sup_{n \geq 0} [\min\{\|T^n x_1\|, \|T^n x_2\|, \dots, \|T^n x_p\|\}] < \infty.$$

If one of the vectors  $x_k$  has a bounded orbit under  $T$  (in particular, if one of the vectors is zero), then clearly, the set  $\{x_k\}_{k=1}^p$  has a jointly bounded orbit. However, it may happen that the set  $\{x_k\}_{k=1}^p$  has a jointly bounded orbit even when each individual vector has an unbounded orbit. This would happen if the orbit of each of the vectors  $x_k$  is unbounded and yet there is an  $M > 0$  such that for each  $n \geq 0$ , there is a  $k$  such that  $\|T^n x_k\| \leq M$ . One can easily construct a pair of vectors  $x_1, x_2 \in \ell^2(\mathbb{N})$  each of which has an unbounded orbit under twice the backward shift, and yet the pair  $\{x_1, x_2\}$  has a jointly bounded orbit. One may also consider, not only finite sets, but also arbitrary sets with jointly bounded orbits. In this case one should replace the minimum in the definition above with an infimum.

See Proposition 6.12 for further examples and an application of the following result.

**Proposition 3.6.** *If  $n \geq 1$  and  $T$  is an  $n$ -weakly hypercyclic operator on a Banach space  $X$ , then there does not exist a set of non-zero vectors with a jointly bounded orbit under  $T^*$  that has cardinality less than or equal to  $n$ .*

**Proof.** Suppose that  $T$  is  $n$ -weakly hypercyclic on  $X$  and, by way of contradiction, suppose that  $S = \{f_1, \dots, f_m\} \subseteq X^*$  is a set of non-zero vectors with a jointly bounded orbit under  $T^*$ , where  $m \leq n$ . Let  $v$  be an  $n$ -weakly hypercyclic vector for  $T$  and define  $F : X \rightarrow \mathbb{F}^m$  by  $F(x) = (f_1(x), f_2(x), \dots, f_m(x))$ . Since  $\text{Orb}(v, T)$  is  $n$ -weakly dense in  $X$ , then we must have that  $F(\text{Orb}(v, T))$  is dense in  $F(X)$  (see Proposition 2.4, item (4)). However, since  $S$  has a jointly bounded orbit under  $T^*$ , say with bound  $M$ , then every vector in  $F(\text{Orb}(v, T))$  has at least one of its coordinates bounded by  $M\|v\|$ . However, one can easily see, since each  $f_k$  is non-zero, that the subspace  $F(X)$  contains vectors all of whose coordinates have absolute value larger than  $M\|v\|$ . To see this, simply choose a vector  $x \in X \setminus \bigcup_{k=1}^m \ker(f_k)$ , then each coordinate of  $F(x)$  is non-zero, hence if  $c > 0$  and large enough, then each coordinate of  $F(cx)$  will have absolute



value larger than  $M\|v\|$ . Thus it follows that  $F(\text{Orb}(v, T))$  is not dense in  $F(X)$ , contradicting the fact that  $\text{Orb}(v, T)$  is  $n$ -weakly dense in  $X$ .  $\square$

#### 4. $n$ -Weakly hypercyclic/supercyclic matrices

It is well known that a  $2 \times 2$  irrational rotation matrix on  $\mathbb{R}^2$  is supercyclic and hence 2-weakly supercyclic. It is also known that there are no supercyclic matrices on  $\mathbb{C}^n$  when  $n \geq 2$  and no supercyclic matrices on  $\mathbb{R}^n$  when  $n \geq 3$ . In [15] Feldman proves the following results, which surprisingly shows that there is some weak dynamics of matrices that exists in higher (even) dimensions!

**Theorem 4.1** (Non-existence results).

- (1) There are no 1-weakly hypercyclic operators on  $\mathbb{R}^n$  or  $\mathbb{C}^n$  for  $n \geq 1$ .
- (2) There are no 2-weakly supercyclic operators on  $\mathbb{C}^n$  for  $n \geq 2$ .
- (3) There are no 3-weakly supercyclic operators on  $\mathbb{R}^n$  for  $n \geq 3$ .
- (4) There are 2-weakly supercyclic operators on  $\mathbb{R}^n$  if and only if  $n$  is even.

Let  $J_1(r, \theta) = \begin{bmatrix} r \cos(\theta) & -r \sin(\theta) \\ r \sin(\theta) & r \cos(\theta) \end{bmatrix}$  be the matrix that rotates by an angle of  $\theta$  and dilates by  $r > 0$ .

**Theorem 4.2** (Existence in even dimensions). If  $T = \bigoplus_{k=1}^p J_1(r, \theta_k)$ ,  $r > 0$ , acting on  $\mathbb{R}^{2p}$  and if the set  $\{\pi, \theta_1, \theta_2, \dots, \theta_p\}$  is linearly independent over the field  $\mathbb{Q}$  of rational numbers, then  $T$  is 2-weakly supercyclic on  $\mathbb{R}^{2p}$ .

See Feldman [15] for these proofs.

#### 5. Towards an $n$ -weak Ansari Theorem

In this section we will show that there are some natural and simple ways for producing a vector whose orbit under twice the backward shift is  $n$ -weakly dense but not  $(n+1)$ -weakly dense. We will then use this to show that the union of two orbits for twice the backward shift may be 1-weakly dense with out either orbit being 1-weakly dense. Also, we will show that if  $T$  is twice the backward shift, then  $T$  and  $T^2$  do not have the same 1-weakly hypercyclic vectors. We will also show that the natural form of Ansari's Theorem is not true in the  $n$ -weak setting. Ansari's Theorem says that if  $x$  is a hypercyclic vector for  $T$ , then  $x$  is also a hypercyclic vector for  $T^p$ . We will show the surprising result that there exist a vector  $x$  and an operator  $T$  such that  $x$  is an  $n$ -weakly hypercyclic vector for  $T$ , but  $x$  is an  $n$ -weakly hypercyclic vector for  $T^p$  if and only if  $p$  and  $(n+1)$  are relatively prime. More generally in Theorem 5.8 we will show that for this  $x$  and  $T$ , the orbit of  $x$  under  $T$  is  $n$ -weakly dense, however the orbit of  $x$  under  $T^p$  is only  $(\frac{n+1}{d} - 1)$ -weakly dense where  $d = \gcd(p, n+1)$ . This suggests a more appropriate form for an  $n$ -weak Ansari Theorem.

In this section  $B$  will denote the (unweighted) unilateral backward shift on  $\ell^2(\mathbb{N})$ . Also for an operator  $T$ ,  $HC(T)$  will denote the set of all hypercyclic vectors for  $T$  and  $WHC_n(T)$  will denote the set of all  $n$ -weakly hypercyclic vectors for  $T$ .

**Proposition 5.1.** Let  $D : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  be the doubling map defined by

$$Dx = (x_0, x_0, x_1, x_1, x_2, x_2, \dots) \quad \text{where } x = (x_n)_{n=0}^\infty \in \ell^2(\mathbb{N}).$$

Then  $D$  is a multiple of an isometry and  $\text{Range}(D) \cup \text{Range}(BD)$  is 1-weakly dense in  $\ell^2(\mathbb{N})$  but not 2-weakly dense in  $\ell^2(\mathbb{N})$  where  $B$  is the backward shift on  $\ell^2(\mathbb{N})$ .

**Proof.** Notice that the (doubling) map  $D : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  described above that “repeats each coordinate twice” is a continuous linear map that satisfies  $\|Dx\|^2 = 2\|x\|^2$  for every  $x \in \ell^2(\mathbb{N})$ . Thus,  $\frac{1}{\sqrt{2}}D$  is an isometry.

Let  $X = \text{Range}(D) = D(\ell^2(\mathbb{N}))$  and  $Y = \text{Range}(BD) = BD(\ell^2(\mathbb{N}))$ . We will show that the set  $X \cup Y$  is 1-weakly dense in  $\ell^2(\mathbb{N})$  but not 2-weakly dense in  $\ell^2(\mathbb{N})$ . Notice that  $X$  and  $Y$  are both subspaces of  $\ell^2(\mathbb{N})$ .

**Claim.**  $X + Y$  is norm dense in  $\ell^2(\mathbb{N})$ .

To verify this claim, let  $\{e_n\}_{n=0}^\infty$  be the standard unit basis vectors for  $\ell^2(\mathbb{N})$  and notice that for every  $n \geq 0$ , we have

$$x_n = (e_{2n} + e_{2n+1}) \in X \quad \text{and} \quad y_n = (e_{2n+1} + e_{2n+2}) \in Y.$$

Thus,

$$x_n - y_n = (e_{2n} - e_{2n+2}) \in X + Y \quad \text{and} \quad x_{n+1} - y_n = (e_{2n+3} - e_{2n+1}) \in X + Y.$$

Thus if  $g = \{\hat{g}(n)\}_{n=0}^\infty \in \ell^2$  is a non-zero linear functional that annihilates the subspace  $X + Y$ , then we must have that  $g(e_{2n}) - g(e_{2n+2}) = 0$  and  $g(e_{2n+3}) - g(e_{2n+1}) = 0$  for all  $n \geq 0$ . Thus,  $\hat{g}(2n) = \hat{g}(2n+2)$  for all  $n \geq 0$  and  $\hat{g}(2n+1) = \hat{g}(2n+3)$  for all  $n \geq 0$ . Since  $g \in \ell^2(\mathbb{N})$  this implies that  $g = 0$ . Thus  $X + Y$  is dense in  $\ell^2(\mathbb{N})$ .

Now since  $X$  and  $Y$  are subspaces of  $\ell^2(\mathbb{N})$  and  $X + Y$  is dense in  $\ell^2(\mathbb{N})$ , then we have that  $X \cup Y$  is closed under scalar multiplication and has dense linear span, thus it follows from Proposition 2.8 that  $X \cup Y$  is 1-weakly dense in  $\ell^2(\mathbb{N})$ .

**Claim.**  $X \cup Y$  is not 2-weakly dense in  $\ell^2(\mathbb{N})$ .

Let  $f = e_0 - e_1 = (1, -1, 0, 0, \dots)$  and let  $g = e_1 - e_2 = (0, 1, -1, 0, 0, \dots)$ . Since  $f$  and  $g$  are independent, then  $F : \ell^2(\mathbb{N}) \rightarrow \mathbb{C}^2$  given by  $F(z) = \begin{bmatrix} f(z) \\ g(z) \end{bmatrix}$  is onto. However, since  $f(X) = \{0\}$  and  $g(Y) = \{0\}$ , it follows that  $F(X \cup Y)$  lies in the union of two proper subspaces of  $\mathbb{C}^2$  and hence is not dense in  $\mathbb{C}^2$ . Thus,  $X \cup Y$  is not 2-weakly dense in  $\ell^2(\mathbb{N})$ .  $\square$

**Proposition 5.2.** If  $D$  is the doubling map on  $\ell^2(\mathbb{N})$  and  $v$  is a hypercyclic vector for  $4B$ , then  $Dv$  is a 1-weakly hypercyclic vector for  $2B$  that is not a 2-weakly hypercyclic vector for  $2B$ .

It follows that the doubling map  $D$  naturally induces a map from  $HC(4B) \rightarrow WHC_1(2B)$ .

**Proof of Proposition 5.2.** Notice that  $DBx = (x_1, x_1, x_2, x_2, \dots) = B^2Dx$  for every  $x = (x_n)_{n=0}^\infty \in \ell^2(\mathbb{N})$ . It follows that

$$DB = B^2D.$$

Thus,

$$D(4B) = (2B)^2D.$$

So, if  $T = 2B$ , then

$$D(4B) = T^2D.$$

Let  $v$  be a hypercyclic vector for  $4B$ , then the previous intertwining relation implies that

$$\text{cl}[\text{Orb}(Dv, T^2)] = D(\text{cl}[\text{Orb}(v, 4B)]) = D(\ell^2(\mathbb{N})). \quad (1)$$

Since  $\{T^n Dv\}_{n=0}^\infty = \{T^{2n} Dv\}_{n=0}^\infty \cup \{T(T^{2n} Dv)\}_{n=0}^\infty$  we get

$$\text{clOrb}(Dv, T) = D(\ell^2(\mathbb{N})) \cup T(D(\ell^2(\mathbb{N}))) = X \cup Y \quad (2)$$

where  $X = \text{Range}(D)$  and  $Y = \text{Range}(BD)$ . It now follows by Proposition 5.1,  $X \cup Y$  is 1-weakly dense in  $\ell^2(\mathbb{N})$  but not 2-weakly dense in  $\ell^2(\mathbb{N})$ .

Thus if  $v \in HC(4B)$ , then  $Dv$  is a 1-weakly hypercyclic vector for  $T = 2B$  that is not a 2-weakly hypercyclic vector.  $\square$

Ansari [1] proved that an operator  $T$  is hypercyclic if and only if  $T^n$  is hypercyclic and that  $T$  and  $T^n$  have the same set of hypercyclic vectors for any positive integer  $n$ . Bourdon and Feldman's [9] Somewhere Dense Theorem applies to operators on locally convex spaces and thus applies to operators on Banach spaces with the weak topology. Thus it follows that  $T$  and  $T^n$  have the same set of weakly hypercyclic vectors. However, below we shall see that  $T$  and  $T^n$  need not have the same set of 1-weakly hypercyclic vectors. In particular,  $T$  and  $T^2$  do not have the same set of 1-weakly hypercyclic vectors.

**Theorem 5.3.** *If  $T$  is twice the backward shift,  $T = 2B$ ,  $v \in HC(4B)$ , and  $x = Dv$  where  $D$  is the doubling map, then  $x$  is a 1-weakly hypercyclic vector for  $T^n$  if and only if  $n$  is an odd positive integer.*

**Proof.** Let  $v \in HC(4B)$  and let  $x = Dv$  where  $D$  is the doubling map. Then by Proposition 5.2 we have that  $\text{clOrb}(x, T^2) = D(\ell^2(\mathbb{N}))$ . Suppose now that  $n$  is even. Then  $\text{clOrb}(x, T^n) \subseteq \text{clOrb}(x, T^2) = D(\ell^2(\mathbb{N}))$ . Since  $D(\ell^2(\mathbb{N}))$  is a proper closed subspace of  $\ell^2(\mathbb{N})$ , then it is not 1-weakly dense in  $\ell^2(\mathbb{N})$ , thus  $x$  is not a 1-weakly hypercyclic vector for  $T^n$  when  $n$  is even.

On the other hand, suppose that  $n$  is odd. Since  $v \in HC(4B)$ , then by Ansari's Theorem  $v \in HC((4B)^n)$ . Also, since

$$D(4B) = (2B)^2D$$

then it follows that

$$D(4B)^n = (2B)^{2n}D.$$

Thus since  $T = 2B$  and since  $D$  is a multiple of an isometry, we have

$$\begin{aligned}\text{cl Orb}(Dv, T^{2n}) &= \text{cl Orb}(Dv, (2B)^{2n}) = \text{cl } D(\text{Orb}(v, (4B)^n)) \\ &= D(\text{cl Orb}(v, (4B)^n)) = D(\ell^2(\mathbb{N})).\end{aligned}$$

It follows that

$$\text{cl Orb}(x, T^{2n}) = D(\ell^2(\mathbb{N})).$$

Since

$$\text{Orb}(x, T^n) = \text{Orb}(x, T^{2n}) \cup T^n(\text{Orb}(x, T^{2n}))$$

by taking closures of the above identity, and since  $D$  is a multiple of an isometry, and using the fact that  $T^p(D(\ell^2(\mathbb{N}))) = D(\ell^2(\mathbb{N}))$  when  $p$  is even and  $T^p(D(\ell^2(\mathbb{N}))) = T(D(\ell^2(\mathbb{N})))$  when  $p$  is odd we get (since  $n$  is odd) that

$$\begin{aligned}\text{cl Orb}(x, T^n) &= D(\ell^2(\mathbb{N})) \cup T^n(D(\ell^2(\mathbb{N}))) = D(\ell^2(\mathbb{N})) \cup T(D(\ell^2(\mathbb{N}))) \\ &= \text{Range}(D) \cup \text{Range}(TD) = \text{Range}(D) \cup \text{Range}(BD).\end{aligned}$$

It now follows from Proposition 5.1 that  $\text{Orb}(x, T^n)$  is 1-weakly dense in  $\ell^2(\mathbb{N})$ , hence  $x$  is a 1-weakly hypercyclic vector for  $T^n$  when  $n$  is odd.  $\square$

Recall that an operator  $T$  on a space  $X$  is *multi-hypercyclic* if there is a finite number vectors  $x_1, \dots, x_n$  in  $X$  such that  $\bigcup_{k=1}^n \text{Orb}(x_k, T)$  is dense in  $X$ . It has been shown independently by Costakis [13], Peris [20], and Bourdon and Feldman [9] that multi-hypercyclic operators are in fact hypercyclic. In fact, if  $\bigcup_{k=1}^n \text{Orb}(x_k, T)$  is dense in  $X$ , then  $\text{Orb}(x_k, T)$  is dense in  $X$  for some  $k$ . In [9] it is also shown that the same result holds for the weak topology. However, below we shall see that the latter result does not hold for the  $n$ -weak “pseudo-topology”.

**Corollary 5.4.** *If  $T = 2B$ , then there exist vectors  $x_1, x_2 \in \ell^2(\mathbb{N})$  such that*

$$\text{Orb}(x_1, T^2) \cup \text{Orb}(x_2, T^2)$$

*is 1-weakly dense in  $\ell^2(\mathbb{N})$ , but neither  $\text{Orb}(x_1, T^2)$  nor  $\text{Orb}(x_2, T^2)$  is 1-weakly dense in  $\ell^2(\mathbb{N})$ .*

**Proof.** As in the proof of Proposition 5.2, let  $v \in HC(4B)$  and let  $x_1 = Dv$  and  $x_2 = BDv$ . Then by Eqs. (1) and (2) from Proposition 5.2 and also using Proposition 5.1 that  $\text{Range}(D) \cup \text{Range}(BD)$  is 1-weakly dense in  $\ell^2$  we see that  $\text{Orb}(x_1, T^2) \cup \text{Orb}(x_2, T^2)$  is 1-weakly dense in  $\ell^2$ , but  $\text{cl Orb}(x_1, T^2) = \text{Range}(D)$  and  $\text{cl Orb}(x_2, T^2) = \text{Range}(BD)$  are both proper closed subspaces, hence not 1-weakly dense in  $\ell^2(\mathbb{N})$ .  $\square$

We now give another method for creating  $n$ -weakly hypercyclic vectors for twice the backward shift that are not  $(n + 1)$ -weakly hypercyclic and also consider a version of Ansari's Theorem for these particular vectors. First we state a simple but useful lemma.

**Lemma 5.5.** *If  $T$  is an operator on a space  $X$  and  $x \in X$  and  $n$  is a positive integer, then*

$$\text{Orb}(x, T) = \bigcup_{r=0}^{n-1} T^r(\text{Orb}(x, T^n)).$$

**Proof.** Note that  $\text{Orb}(x, T) = \{T^k x\}_{k=0}^\infty$ . Since  $n$  is given and fixed, every non-negative integer  $k$  may be written as  $k = nq + r$  where  $q \geq 0$  and  $0 \leq r \leq (n - 1)$  where  $q$  and  $r$  are also integers. Thus we have

$$\begin{aligned} \text{Orb}(x, T) &= \{T^k x\}_{k=0}^\infty = \{T^{nq+r} x\}_{q=0, r=0}^{\infty, n-1} \\ &= \{T^{nq} x\}_{q=0}^\infty \cup \{T(T^{nq} x)\}_{q=0}^\infty \cup \{T^2(T^{nq} x)\}_{q=0}^\infty \cup \dots \cup \{T^{n-1}(T^{nq} x)\}_{q=0}^\infty \\ &= \text{Orb}(x, T^n) \cup T(\text{Orb}(x, T^n)) \cup T^2(\text{Orb}(x, T^n)) \cup \dots \cup T^{n-1}(\text{Orb}(x, T^n)) \\ &= \bigcup_{r=0}^{n-1} T^r(\text{Orb}(x, T^n)). \quad \square \end{aligned}$$

**Proposition 5.6.** *If  $n \geq 1$ , and if  $v = (v_k)_{k=0}^\infty$  is a hypercyclic vector for  $2^{n+1}B^n$  and we let*

$$Z_n v = (0, v_0, v_1, \dots, v_{n-1}, 0, v_n, v_{n+1}, \dots, v_{2n-1}, 0, v_{2n}, \dots, v_{3n-1}, 0, v_{3n}, \dots)$$

*then  $\text{Orb}(Z_n v, 2B)$  is  $n$ -weakly dense but not  $(n + 1)$ -weakly dense in  $\ell^2(\mathbb{N})$ . In particular, if  $v \in HC(2^{n+1}B^n)$ , then  $Z_n v \in WHC_n(2B)$ .*

**Proof.** For any  $x \in \ell^2(\mathbb{N})$ , let  $Z_n x$  be obtained by inserting zeros into the vector  $x$  having  $n$  terms of  $x$  between consecutive zeros, as above and the initial term being zero. Thus,

$$(Z_n x)_{k(n+1)} = 0 \quad \text{for all } k \geq 0 \quad \text{and} \quad (Z_n x)_{k(n+1)+j+1} = x_{kn+j} \quad \text{for } k \geq 0, 0 \leq j \leq n-1.$$

It is easy to see that  $T^{n+1}Z_n x$  also has zeros in coordinates with positions  $k(n + 1)$ ,  $k \geq 0$ . Notice that  $Z_n$  is an (into) isometry from  $\ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  and satisfies

$$Z_n(B^n x) = (0, x_n, x_{n+1}, \dots, x_{2n-1}, 0, x_{2n}, \dots) = B^{n+1}(Z_n x) \quad \text{for all } x \in \ell^2(\mathbb{N}).$$

It follows that

$$Z_n(2^{n+1}B^n x) = (2B)^{n+1}(Z_n x) \quad \text{for all } x \in \ell^2(\mathbb{N}).$$

Thus if  $T = 2B$ , then  $Z_n(2^{n+1}B^n x) = T^{n+1}(Z_n x)$  for all  $x \in \ell^2(\mathbb{N})$ . So if  $v$  is a hypercyclic vector for  $2^{n+1}B^n$ , then since  $Z_n$  is an isometry we will have that

$$\text{cl}[\text{Orb}(Z_n v, T^{n+1})] = Z_n(\ell^2(\mathbb{N})).$$

That is, the closure of the orbit of  $Z_nv$  under the operator  $T^{n+1}$  is the set of all vectors having zeros in the “correct” positions  $k(n+1)$ ,  $k \geq 0$ :

$$\text{cl Orb}(Z_nv, T^{n+1}) = Z_n(\ell^2(\mathbb{N})) = \{x = (x_k)_{k=0}^\infty \in \ell^2(\mathbb{N}) : x_{k(n+1)} = 0 \text{ for all } k \geq 0\}.$$

Thus by Lemma 5.5 we have

$$\text{cl Orb}(Z_nv, T) = \bigcup_{k=0}^n T^k Z_n(\ell^2(\mathbb{N})).$$

In particular,  $\text{cl}[\text{Orb}(Z_nv, T)]$  contains the set

$$X_n = \{x = (x_k)_{k=0}^\infty \in \ell^2(\mathbb{N}) : x_k = 0 \text{ for all but at most } n \text{ coordinates}\}.$$

It follows from Corollary 2.10 that  $X_n$ , and hence  $\text{Orb}(Z_nv, T)$ , is  $n$ -weakly dense in  $\ell^2(\mathbb{N})$ .

**Claim.**  $\text{Orb}(Z_nv, T)$  is not  $(n+1)$ -weakly dense in  $\ell^2(\mathbb{N})$ .

Consider the subspace  $\mathcal{M}$  spanned by  $\{e_0, e_1, \dots, e_n\}$ . We claim that the orthogonal projection of  $\text{Orb}(Z_nv, T)$  onto  $\mathcal{M}$  is not dense in  $\mathcal{M}$ . This is simply because any  $(n+1)$  consecutive coordinates of the vector  $Z_nv$  will contain a zero. Thus the orthogonal projection of the orbit of  $Z_nv$  onto  $\mathcal{M}$  will consist of vectors each of which will have at least one coordinate equal to zero. Thus the projection of  $\text{Orb}(Z_nv, T)$  is contained in the union of  $(n+1)$  hyperplanes in  $\mathcal{M}$  and hence not dense in  $\mathcal{M}$ . Thus,  $\text{Orb}(Z_nv, T)$  is not  $(n+1)$ -weakly dense in  $\ell^2(\mathbb{N})$ .  $\square$

**Lemma 5.7.** *If  $a$  and  $b$  are positive integers and  $d$  is the greatest common divisor of  $a$  and  $b$ ,  $d = \gcd(a, b)$ , then there exist positive integers  $x, y$  such that  $ax - by = d$ . Furthermore,  $d = \min\{k \in \mathbb{N} : k = ax - by \text{ for some } x, y \in \mathbb{N}\}$ .*

**Proof.** It is well known that if  $d = \gcd(a, b)$ , then there exist integers  $n, k$  (positive or negative) such that  $an + bk = d$  and that  $d = \min\{k \in \mathbb{N} : k = ax + by \text{ for some } x, y \in \mathbb{Z}\}$ . However, notice that we also have

$$a(n + bj) + b(k - aj) = d$$

where  $j$  is any integer. Thus if we choose  $j$  large enough and let  $x = n + bj$  and  $y = aj - k$ , then both  $x$  and  $y$  will be positive integers and we will have  $d = ax - by$ .  $\square$

It is easy to check that if  $c > 1$  and  $B$  is the backward shift and  $n$  is a positive integer, then  $cB^n$  is a hypercyclic operator (it satisfies the hypercyclicity criterion).

**Theorem 5.8.** *Let  $T = 2B$  be twice the backward shift. Also let  $n$  and  $p$  be positive integers,  $d = \gcd(n+1, p)$ ,  $v = (v_k)_{k=0}^\infty$  a hypercyclic vector for  $2^{n+1}B^n$  and let*

$$x = Z_nv = (0, v_0, v_1, \dots, v_{n-1}, 0, v_n, v_{n+1}, \dots, v_{2n-1}, 0, v_{2n}, \dots, v_{3n-1}, 0, v_{3n}, \dots),$$

*then the following hold:*

- (i)  $\text{Orb}(x, T)$  is  $n$ -weakly dense in  $\ell^2(\mathbb{N})$ .
- (ii)  $\text{Orb}(x, T^p)$  is  $(\frac{n+1}{d} - 1)$ -weakly dense in  $\ell^2(\mathbb{N})$  but not  $\frac{n+1}{d}$ -weakly dense.
- (iii)  $\text{Orb}(x, T^p)$  is  $n$ -weakly dense in  $\ell^2(\mathbb{N})$  if and only if  $\gcd(p, n+1) = 1$ .

**Proof.** (i) This follows from Proposition 5.6.

(ii) We know from the proof of Proposition 5.6 that

$$Z_n(2^{n+1}B^n) = (2B)^{n+1}Z_n \quad (1)$$

and since  $x$  is a hypercyclic vector for  $2^{n+1}B^n$ , then we get

$$\text{cl Orb}(x, T^{n+1}) = Z_n(\ell^2(\mathbb{N})). \quad (2)$$

If  $p$  is also a positive integer, then from Eq. (1) we get

$$Z_n(2^{n+1}B^n)^p = (2B)^{(n+1)p}Z_n. \quad (3)$$

Since  $x \in HC(2^{n+1}B^n)$ , then by Ansari's Theorem,  $x \in HC((2^{n+1}B^n)^p)$ . Thus it follows from (3) that

$$\text{cl Orb}(x, T^{(n+1)p}) = Z_n(\ell^2(\mathbb{N})).$$

Since  $\text{Orb}(x, T^p) = \{T^{pk}x\}_{k=0}^\infty \supseteq \{T^{p(n+1)k}\}_{k=0}^\infty = \text{Orb}(x, T^{p(n+1)})$ , we have

$$\text{cl Orb}(x, T^p) \supseteq Z_n(\ell^2(\mathbb{N})). \quad (4)$$

Now by applying Proposition 5.6 to  $T^p$  and with  $(n+1)$  in place of  $n$  we get

$$\begin{aligned} \text{cl Orb}(x, T^p) &= \bigcup_{k=0}^n T^{pk} \text{cl Orb}(x, T^{p(n+1)}) = \bigcup_{k=0}^n T^{pk} Z_n(\ell^2(\mathbb{N})) \\ &= \bigcup_{k=0}^n B^{pk} Z_n(\ell^2(\mathbb{N})) = \bigcup_{k=0}^\infty B^{pk} Z_n(\ell^2(\mathbb{N})). \end{aligned} \quad (5)$$

The last equality follows from the fact that  $B^{n+1}Z_n(\ell^2(\mathbb{N})) = Z_n(\ell^2(\mathbb{N}))$ , which in turn follows from the description below of  $Z_n(\ell^2(\mathbb{N}))$ .

Recall that  $Z_n(\ell^2(\mathbb{N}))$  is the subspace of  $\ell^2(\mathbb{N})$  of all vectors that have zeros in positions  $(n+1)j$  where  $j \geq 0$ . Notice that  $B^k Z_n(\ell^2(\mathbb{N}))$  is the subspace obtained from  $Z_n(\ell^2(\mathbb{N}))$  by shifting all the vectors in  $Z_n(\ell^2(\mathbb{N}))$   $k$  positions to the left. Thus the subspace  $B^k Z_n(\ell^2(\mathbb{N}))$  consists of all vectors in  $\ell^2(\mathbb{N})$  that have zeros in positions  $(n+1)j - k$  where  $j \geq 0$ . Thus from Eq. (5) we have that  $\text{cl Orb}(x, T^p)$  consists of all vectors  $x$  in  $\ell^2(\mathbb{N})$  such that there exists an integer  $k \geq 0$  so that  $x$  has zeros in positions  $\{(n+1)j - pk : j \geq 0\}$ . So,

$$\text{cl Orb}(x, T^p) = \{x \in \ell^2(\mathbb{N}) : \exists k \geq 0 \text{ s.t. } x_{(n+1)j - pk} = 0 \forall j \geq 0\}$$



where we understand (define) that  $x_m = 0$  for all  $m < 0$ , since  $x \in \ell^2(\mathbb{N})$ . Now if we let  $d = \min\{(n+1)j - pk: j, k \geq 0 \text{ and } (n+1)j - pk > 0\}$ , then by Lemma 5.7 we know that  $d = \gcd(a, b)$ . We also know that all integer multiples of  $d$  generate the subgroup  $\{(n+1)j - pk: j, k \in \mathbb{Z}\}$ ; thus all positive multiples of  $d$  are equal to the positive numbers in  $\{(n+1)j - pk: j, k \geq 0\}$ . Thus we have

$$\text{cl Orb}(x, T^p) = \{x \in \ell^2(\mathbb{N}): \exists k \geq 0 \text{ s.t. } x_{kd+(n+1)j} = 0 \forall j \in \mathbb{Z}\}. \quad (6)$$

In order to show that  $\text{Orb}(x, T^p)$  is  $(\frac{n+1}{d} - 1)$ -weakly dense in  $\ell^2(\mathbb{N})$ , we will show that  $\text{cl Orb}(x, T^p)$  contains the set  $X_{(\frac{n+1}{d}-1)}$  consisting of all vectors in  $\ell^2(\mathbb{N})$  with at most  $(\frac{n+1}{d} - 1)$  non-zero coordinates. Since Corollary 2.10 tells us that  $X_{(\frac{n+1}{d}-1)}$  is  $(\frac{n+1}{d} - 1)$ -weakly dense in  $\ell^2(\mathbb{N})$ , then  $\text{cl Orb}(x, T^p)$  will also be  $(\frac{n+1}{d} - 1)$ -weakly dense in  $\ell^2(\mathbb{N})$ .

Let  $y = (y_j)_{j=0}^\infty \in X_{(\frac{n+1}{d}-1)}$ . So,  $y$  has at most  $(\frac{n+1}{d} - 1)$  of its coordinates non-zero. Say that  $y_{j_1}, \dots, y_{j_m}$  are the non-zero coordinates of  $y$  where  $m \leq (\frac{n+1}{d} - 1)$ . Consider the residue classes modulo  $(n+1)$  of the positions of these coordinates:  $[j_1]_{n+1}, [j_2]_{n+1}, \dots, [j_m]_{n+1}$ . There are at most  $m$  distinct residue classes in this list and  $m \leq \frac{n+1}{d} - 1$ . Furthermore, there are  $\frac{n+1}{d}$  distinct residue classes (mod  $(n+1)$ ) of the numbers  $\{kd: k \geq 0\}$  (since  $\frac{n+1}{d}$  is the order of the quotient group  $\mathbb{Z}_{n+1}/\langle d \rangle$ ). That is,  $\{[kd]_{(n+1)}: k \geq 0\}$  has cardinality  $\frac{n+1}{d}$ . Thus, there exists a  $k_0 \in \mathbb{N}$  such that  $[k_0d]_{(n+1)} \neq [j_k]_{(n+1)}$  for any  $k \in \{1, \dots, m\}$ . Then by the definition of the positions  $\{j_1, \dots, j_m\}$  we have that  $y_j = 0$  for all  $j \in [k_0d]_{(n+1)}$ . Or, in other words,  $y_{k_0d+(n+1)j} = 0$  for all  $j \in \mathbb{Z}$ . Thus by (6) we have that  $y \in \text{cl Orb}(x, T^p)$ . It now follows that  $X_{(\frac{n+1}{d}-1)} \subseteq \text{cl Orb}(x, T^p)$ . Thus,  $\text{cl Orb}(x, T^p)$  is  $(\frac{n+1}{d} - 1)$ -weakly dense in  $\ell^2(\mathbb{N})$ .

To see that  $\text{cl Orb}(x, T^p)$  is not  $(\frac{n+1}{d})$ -weakly dense in  $\ell^2(\mathbb{N})$  we will find a subspace of  $\ell^2(\mathbb{N})$  with dimension  $\frac{n+1}{d}$  so that the orthogonal projection of  $\text{cl Orb}(x, T^p)$  onto this subspace does not have dense range.

Let  $q = \frac{n+1}{d}$ . As mentioned above there are  $q$  residue classes mod  $(n+1)$  of the multiples of  $d$ . In fact, they are as follows:  $\{[d]_{n+1}, [2d]_{n+1}, [3d]_{n+1}, \dots, [qd]_{n+1}\}$ . Thus if  $\mathcal{M}$  is the subspace of  $\ell^2(\mathbb{N})$  spanned by the basis vectors  $\{e_{kd}: 1 \leq k \leq q\}$ , then  $\mathcal{M}$  has dimension equal to  $q = \frac{n+1}{d}$ , but the projection of  $\text{cl Orb}(x, T^p)$  onto  $\mathcal{M}$  is not onto. This follows because Eq. (6) above implies that every vector in  $\text{cl Orb}(x, T^p)$  must have a zero in at least one of the positions  $\{kd: 1 \leq k \leq q\}$ . Since  $\text{cl Orb}(x, T^p)$  does not have a dense projection onto  $\mathcal{M}$ , then  $\text{cl Orb}(x, T^p)$  is not  $\frac{n+1}{d}$ -weakly dense. The theorem now follows.  $\square$

Based on the previous theorem, it is natural to ask if  $T$  is an operator on a space  $X$  and  $\text{Orb}(x, T)$  is  $n$ -weakly dense in  $X$ , then is  $\text{Orb}(x, T^p)$   $(\frac{n+1}{d} - 1)$ -weakly dense in  $X$  where  $d = \gcd(p, n+1)$ ? Or is the special case where  $d = 1$  true? That is if  $\text{Orb}(x, T)$  is  $n$ -weakly dense in  $X$  and  $p$  is relatively prime with  $(n+1)$ , then is  $\text{Orb}(x, T^p)$  also  $n$ -weakly dense in  $X$ ?

## 6. Constructing $n$ -weakly hypercyclic operators

A natural area to look for some weak forms of hypercyclicity to exist is the direct sum of a collection of hypercyclic operators. The direct sum may be hypercyclic, but it may not be. Even when it is not hypercyclic, we will show that it can be  $n$ -weakly hypercyclic for some  $n$ . For backward unilateral weighted shifts we can characterize precisely when the direct sum of

such shifts is  $n$ -weakly hypercyclic. This will allow us to show that for every  $n \geq 1$  there are very simple and natural operators that are  $n$ -weakly hypercyclic but not  $(n + 1)$ -weakly hypercyclic.

More generally, in this section we will show that if  $T_1, \dots, T_p$  are operators that each satisfies a strong form of the hypercyclicity criterion, and if  $1 \leq n < p$  and if the direct sum of any  $n$  of the given operators is hypercyclic, then the direct sum of all of the operators is  $n$ -weakly hypercyclic.

To begin we need the following simple variation on the well-known concept of transitivity.

**Lemma 6.1.** *If  $\{f_n\}_{n=1}^\infty$  is a sequence of continuous functions on a complete separable metric space  $X$ ,  $f_n : X \rightarrow X$ , and  $K \subseteq X$  is a set and if for any two non-empty open sets  $U$  and  $V$  in  $X$  with  $V \cap K \neq \emptyset$ , there exists an  $n \geq 1$  such that  $f_n(U) \cap V \neq \emptyset$ , then there is a dense  $G_\delta$  set  $\Omega \subseteq X$  such that for every  $x \in \Omega$ ,  $K \subseteq \text{cl}\{f_n(x)\}_{n=1}^\infty$ .*

**Proof.** Let  $\mathcal{V}$  be a countable basis of open sets for  $X$  and let  $\{V_j\}_{j=1}^\infty$  be an enumeration of all the elements of  $\mathcal{V}$  that have a non-empty intersection with  $K$ . Our hypothesis guarantees that for each  $j \geq 1$ ,  $\bigcup_{n=1}^\infty (f_n)^{-1}(V_j)$  is a dense open set in  $X$ . Hence,  $\Omega := \bigcap_{j=1}^\infty \bigcup_{n=1}^\infty (f_n)^{-1}(V_j)$  is a dense  $G_\delta$  set in  $X$  with the required property.  $\square$

The following is the well-known hypercyclicity criterion; it was shown by Bes and Peris [7] that an operator  $T$  satisfies the hypercyclicity criterion if and only if  $T \oplus T$  is hypercyclic.

**Definition 6.2.** An operator  $T$  on a separable Banach space  $X$  is said to satisfy the *hypercyclicity criterion* if there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, two dense sets  $D_1, D_2 \subseteq X$ , and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1)  $T^{n_k}x \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x \in D_1$ .
- (2)  $S_{n_k}y \rightarrow 0$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .
- (3)  $T^{n_k}S_{n_k}y \rightarrow y$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .

More accurately, we say that  $T$  satisfies the hypercyclicity criterion with respect to the sequence  $\{n_k\}$  if the above conditions hold.

It is easy to show (see the proof below) that if  $T$  satisfies the hypercyclicity criterion, then there exists a dense  $G_\delta$  set  $\Omega \subseteq X$  such that for each  $x \in \Omega$ ,  $\text{cl}[\text{Orb}(x, T)] = X$ .

We now establish a generalization of the hypercyclicity criterion, which we call the containment criterion. When  $D_2$  is dense in  $X$  (or somewhere dense), the containment criterion becomes a hypercyclicity criterion. In fact, if  $D_2$  is required to be dense, then it is equivalent to the hypercyclicity criterion (via the 3-open set condition, see [5, p. 81 and p. 27]). However in what follows it is important that  $D_2$  need not be dense.

**Theorem 6.3 (Containment criterion).** *If  $T$  is an operator on a separable Banach space  $X$  and if there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, a dense set  $D_1 \subseteq X$ , and another (not necessarily dense) set  $D_2 \subseteq X$  and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:*

- (1)  $T^{n_k}x \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x \in D_1$ ;  
 (2) for each  $y \in D_2$ , there exists a subsequence  $\{n_{k_j}\}_{j=1}^\infty$  of  $\{n_k\}_{k=1}^\infty$  such that  $S_{n_{k_j}}y \rightarrow 0$  and  $T^{n_{k_j}}S_{n_{k_j}}y \rightarrow y$  as  $j \rightarrow \infty$ ,

then there exists a dense  $G_\delta$  set  $\Omega \subseteq X$  such that for each  $x \in \Omega$ ,  $D_2 \subseteq \text{cl}[\text{Orb}(x, T)]$ .

**Proof.** We will apply Lemma 6.1. If  $U$  and  $V$  are any two open sets with  $V \cap D_2 \neq \emptyset$ , then let  $y \in V \cap D_2$  and since  $D_1$  is dense, we have  $U \cap D_1 \neq \emptyset$ , so choose an  $x \in U \cap D_1$ . By property (2), there is a subsequence  $\{n_{k_j}\}$  with the stated properties. Then for large  $j$ ,  $(x + S_{n_{k_j}}y)$  belongs to  $U$  and  $T^{n_{k_j}}(x + S_{n_{k_j}}y) = T^{n_{k_j}}x + T^{n_{k_j}}S_{n_{k_j}}y$  will belong to  $V$ . The theorem now follows by Lemma 6.1.  $\square$

If the set  $D_2$  above is  $n$ -weakly dense in  $X$ , then the above theorem implies that  $T$  would be  $n$ -weakly hypercyclic with a dense  $G_\delta$  set of  $n$ -weakly hypercyclic vectors. We illustrate this with an example.

If  $T_1$  is a unilateral backward weighted shift on  $\ell^2(\mathbb{N})$ :  $T_1 e_n = w_n e_{n-1}$  for  $n \geq 1$  and  $T_1 e_0 = 0$ , then  $T_1$  is hypercyclic if and only if  $\sup_n p_n = \infty$  where  $p_n = w_1 w_2 \cdots w_n$ . If  $T_2$  is another unilateral backward weighted shift with weights  $\{\omega_n\}$ , then  $T_1 \oplus T_2$  is hypercyclic if and only if  $\sup_n \min\{p_n, q_n\} = \infty$  where  $p_n$  is as above and  $q_n = \omega_1 \omega_2 \cdots \omega_n$ . Thus it is easy to construct two weighted shifts, both of which are hypercyclic, but so that their direct sum is not hypercyclic.

**Example 6.4.** If  $T_1$  and  $T_2$  are both hypercyclic unilateral backward weighted shifts on  $\ell^2(\mathbb{N})$ , then  $T = T_1 \oplus T_2$  is 1-weakly hypercyclic on  $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ .

**Proof.** Let  $\mathcal{F}$  be the set of all vectors in  $\ell^2(\mathbb{N})$  with finite support, that is, vectors with at most finitely many non-zero coordinates and let  $\mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ . Also let

$$D_1 = \mathcal{F} \oplus \mathcal{F} = \{(x, y) \in \mathcal{H}: x, y \in \mathcal{F}\}$$

and

$$D_2 = (\mathcal{F} \oplus \{0\}) \cup (\{0\} \oplus \mathcal{F}).$$

Clearly, then  $D_1$  is dense in  $\mathcal{H}$  and  $T^n v \rightarrow 0$  as  $n \rightarrow \infty$  for all  $v \in D_1$  (in fact if  $v \in D_1$ , then  $T^n v = 0$  for all large  $n$ ). Since  $T_1$  is a hypercyclic backward unilateral weighted shift it satisfies the hypercyclicity criterion and thus there is a sequence  $\{n_{k,1}\}_{k=1}^\infty$  and functions  $S_{k,1} : \mathcal{F} \rightarrow \ell^2(\mathbb{N})$  such that

$$S_{k,1}y \rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ for each } y \in \mathcal{F} \quad \text{and} \\ T_1^{n_{k,1}} S_{k,1}y = y \quad \text{for each } y \in \mathcal{F}.$$

Also since  $T_2$  is hypercyclic, it also satisfies the hypercyclicity criterion so there is another sequence  $\{n_{k,2}\}_{k=1}^\infty$  and functions  $S_{k,2} : \mathcal{F} \rightarrow \ell^2(\mathbb{N})$  such that

$$S_{k,2}y \rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ for each } y \in \mathcal{F} \quad \text{and} \\ T_2^{n_{k,2}} S_{k,2}y = y \quad \text{for each } y \in \mathcal{F}.$$

By passing to subsequences if necessary, we may suppose that the two sequences  $\{n_{k,1}\}_{k=1}^\infty$  and  $\{n_{k,2}\}_{k=1}^\infty$  are disjoint. Let  $\{n_k\}$  be an increasing enumeration of  $\{n_{k,1}\}_{k=1}^\infty \cup \{n_{k,2}\}_{k=1}^\infty$  and define  $S_k := S_{k,1} \oplus S_{k,2} : D_2 \rightarrow \mathcal{H}$ . Then one can easily check that  $T^{n_k} S_k v = v$  for all  $v \in D_2$  and if  $v = (x, y) \in D_2$ , then there is a subsequence of  $\{n_k\}$ , namely  $\{n_{k,1}\}$  or  $\{n_{k,2}\}$  depending on whether  $x = 0$  or  $y = 0$ , so that  $S_{n_{k,j}} v \rightarrow 0$  as  $k \rightarrow \infty$  (where  $j = 1$  or  $2$  depending on  $v$ ). It now follows from Theorem 6.3 that there is a dense  $G_\delta$  set  $\Omega \subseteq \mathcal{H}$  such that  $D_2 \subseteq \text{cl}[\text{Orb}(v, T)]$  for each  $v \in \Omega$ . Now since  $D_2$  has dense linear span in  $\mathcal{H}$  and is closed under scalar multiplication, then by Proposition 2.8 the set  $D_2$  is 1-weakly dense in  $\mathcal{H}$ . It follows that  $T$  is 1-weakly hypercyclic, as desired.  $\square$

The proof above can clearly be adapted to prove the following.

**Corollary 6.5.** *If  $\{T_n\}_{n \in J}$  is any finite or infinite sequence of (uniformly bounded) hypercyclic backward unilateral weighted shifts, and if  $T = \bigoplus_{n \in J} T_n$ , then  $T$  is 1-weakly hypercyclic.*

The techniques above can also be applied to many more hypercyclic operators than just backward unilateral weighted shifts as we will now see and as one can see from the proof of the previous example. One of the key properties of these shifts is that  $T^n x \rightarrow 0$  for a dense set of  $x$ 's. We now introduce a stronger form of the hypercyclicity criterion; namely it has a stronger hypothesis, from which we will get a stronger conclusion (namely Theorem 6.10).

**Definition 6.6.** An operator  $T$  on a separable Banach space  $X$  is said to satisfy the *strong hypercyclicity criterion* if there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, two dense sets  $D_1, D_2 \subseteq X$ , and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1)  $T^n x \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in D_1$ .
- (2)  $S_{n_k} y \rightarrow 0$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .
- (3)  $T^{n_k} S_{n_k} y \rightarrow y$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .

More accurately, we say that  $T$  satisfies the strong hypercyclicity criterion with respect to the sequence  $\{n_k\}$  if the above conditions hold.

**Remark.** Notice that the only difference between the strong hypercyclicity criterion and the hypercyclicity criterion (see Definition 6.2) is in condition (1). For the hypercyclicity criterion, condition (1) would be replaced with  $T^{n_k} x \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x \in D_1$ . However, for the strong hypercyclicity criterion we require the stronger result that the full orbit goes to zero on a dense set.

**Lemma 6.7.** *If  $T_1, T_2, \dots, T_n$  are operators each satisfying the strong hypercyclicity criterion and if their direct sum  $T = T_1 \oplus T_2 \oplus \dots \oplus T_n$  is hypercyclic, then  $T$  also satisfies the strong hypercyclicity criterion.*

*In other words, if the direct sum is hypercyclic, then there is a common sequence  $\{n_k\}$  such that each of the operators  $T_1, \dots, T_n$  satisfies the strong hypercyclicity criterion with respect to the sequence  $\{n_k\}$ .*

**Proof.** Since  $T$  is hypercyclic and has a dense set of vectors whose orbits converge to zero, it follows that  $T$  satisfies the hypercyclicity criterion with respect to some sequence  $\{n_k\}$  (see [6, Corollary 3.2]). Since the only difference between the hypercyclicity criterion and the strong hypercyclicity criterion is having a dense set on which orbits converge to zero (and  $T$  has such a dense set), it follows that  $T$  also satisfies the strong hypercyclicity criterion.  $\square$

Let us set some notation for the following result. Let  $X_1, X_2, \dots, X_p$  be locally convex spaces,  $X = (\bigoplus_{j=1}^p X_j)$ , and let  $\pi_j : X \rightarrow X_j$  be the natural coordinate projection map. Also, if  $J \subseteq \{1, 2, \dots, p\}$ , then define  $\pi_J : X \rightarrow \bigoplus_{j \in J} X_j$  to be the natural coordinatewise projection map. For convenience, let  $X_J = \bigoplus_{j \in J} X_j$ . Notice that  $X_J$  is not actually a subset of  $X$ , but let  $\hat{X}_J$  be that subspace of  $X$  such that  $\pi_J(\hat{X}_J) = X_J$  and  $\pi_{J^C}(\hat{X}_J) = \{0\}$ , where  $J^C = \{1, \dots, p\} \setminus J$ . In other words,

$$\hat{X}_J = \{x \in X : x_j = 0 \text{ for all } j \notin J\}.$$

So if  $1 \leq n < p$  and  $J \subseteq \{1, 2, \dots, p\}$  with  $|J| = n$ , then vectors in  $\hat{X}_J$  have at most  $n$  non-zero coordinates and the “ $J$ -coordinates” of vectors in  $\hat{X}_J$  can be arbitrarily prescribed and the “ $J^C$ -coordinates” of vectors in  $\hat{X}_J$  are all zero. Now define

$$\mathcal{K}_n = \bigcup \{\hat{X}_J : J \subseteq \{1, \dots, n\} \text{ with } |J| = n\}.$$

The set  $\mathcal{K}_n$  is like a union of “coordinate planes”. The following proposition is a natural generalization of Proposition 2.9 and Corollary 2.10.

**Proposition 6.8.** *Let  $\{X_j\}_{j=1}^p$  be locally convex spaces and let  $X = (\bigoplus_{j=1}^p X_j)$ . If  $1 \leq n < p$  and  $\mathcal{K}_n$  is the subset of  $X$  consisting of all vectors in  $X$  with at most  $n$  non-zero coordinates, that is,*

$$\mathcal{K}_n = \{(x_j)_{j=1}^p \in X : |\{j : x_j \neq 0\}| \leq n\}$$

*then  $\mathcal{K}_n$  is  $n$ -weakly dense in  $X$ .*

**Proof.** Suppose that  $F : X \rightarrow \mathbb{F}^n$  is an onto linear operator. Then we must show that  $F(\mathcal{K}_n)$  is dense in  $\mathbb{F}^n$ . In fact, we will show that  $F(\mathcal{K}_n) = \mathbb{F}^n$ . Define linear maps  $F_j : X_j \rightarrow \mathbb{F}^n$  as follows:  $F_j(x) = F(0, 0, \dots, 0, x, 0, \dots, 0)$  where the  $x$  is in the  $j$ th coordinate. Clearly, then  $F_j : X_j \rightarrow \mathbb{F}^n$  is a linear operator such that if  $x = (x_1, \dots, x_p) \in X$ , then  $F(x) = \sum_{j=1}^p F_j(x_j)$ . In particular,  $F(X) = \sum_{j=1}^p F_j(X_j)$ , where the latter sum is the usual sum of subspaces. Since  $F$  maps  $X$  onto  $\mathbb{F}^n$ , we have that

$$\mathbb{F}^n = F(X) = \sum_{j=1}^p F_j(X_j).$$

Thus the collection of subspaces  $\{F_j(X_j)\}_{j=1}^p$  of  $\mathbb{F}^n$  form a spanning set for  $\mathbb{F}^n$ , and hence there is a subcollection of  $n$  of these subspaces that will span  $\mathbb{F}^n$ . Say,  $\{F_j(X_j) : j \in \{j_1, \dots, j_n\}\}$  span  $\mathbb{F}^n$ . If we let  $J = \{j_1, j_2, \dots, j_n\}$ , then  $|J| = n$  and  $\hat{X}_J$  as defined above, consists of all

vectors in  $X$  whose  $J$ -coordinates may be arbitrarily prescribed and whose other coordinates are all zero. It then follows that,  $F(\hat{X}_J) = \sum_{j \in J} F_j(X_j) = \text{span}\{F_j(X_j)\}_{j \in J} = \mathbb{F}^n$ . Since  $\hat{X}_J \subseteq \mathcal{K}_n$  we have that  $\mathbb{F}^n = F(\hat{X}_J) \subseteq F(\mathcal{K}_n)$ . Thus,  $F(\mathcal{K}_n) = \mathbb{F}^n$  as desired.  $\square$

The proof of the following corollary uses the previous proposition and is almost identical to the proof of Corollary 2.10, we leave the details to the reader.

**Corollary 6.9.** *Let  $\{X_j\}_{j=1}^\infty$  be locally convex spaces and let  $X = (\bigoplus_{j=1}^\infty X_j)$ . Suppose that  $X$  is given as a topology such that  $X$  becomes a locally convex space where the coordinate projection maps are all continuous and the set of vectors with at most finitely many non-zero coordinates form a dense set in  $X$ . If  $1 \leq n < \infty$  and  $\mathcal{K}_n$  is the subset of  $X$  consisting of all vectors in  $X$  with at most  $n$  non-zero coordinates, that is,*

$$\mathcal{K}_n = \{(x_j)_{j=1}^\infty \in X : |\{j : x_j \neq 0\}| \leq n\}$$

*then  $\mathcal{K}_n$  is  $n$ -weakly dense in  $X$ .*

The following theorem is a generalization of Example 6.4.

**Theorem 6.10** (Direct sums are  $n$ -weakly hypercyclic). *Suppose that  $T_j \in \mathcal{B}(X_j)$ ,  $1 \leq j \leq p$ , are bounded linear operators on separable Banach spaces each satisfying the strong hypercyclicity criterion. Suppose that  $1 \leq n < p$  and that the direct sum of any  $n$  of the operators  $\{T_j\}_{j=1}^p$  is hypercyclic, then the direct sum  $T = \bigoplus_{j=1}^p T_j$  is  $n$ -weakly hypercyclic.*

**Proof.** Let  $X = (\bigoplus_{j=1}^p X_j)$  and  $T = \bigoplus_{j=1}^p T_j$  and let  $1 \leq n < p$ . We will show that  $T$  satisfies the hypothesis of Theorem 6.3 where  $D_2$  is an  $n$ -weakly dense set of the form of  $\mathcal{K}_n$  as described in Proposition 6.8.

Since each operator  $T_j$ ,  $1 \leq j \leq p$ , satisfies the strong hypercyclicity criterion, there is a dense set  $D_{1,j}$  in  $X_j$  such that  $T_j^n x \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in D_{1,j}$ . Let  $D_1 = \bigoplus_{j=1}^p D_{1,j}$ . Then  $D_1$  is dense in  $X$  and  $T^n x \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in D_1$ . Now we need a set  $D_2$  for  $T$ .

If  $J \subseteq \{1, 2, \dots, p\}$  is a set of cardinality  $n$ ,  $|J| = n$ , then let  $X_J = \bigoplus_{j \in J} X_j$  and  $T_J = \bigoplus_{j \in J} T_j$ . Also let  $\pi_J : X \rightarrow X_J$  be the natural coordinate projection map. For each set  $J \subseteq \{1, \dots, p\}$  with  $|J| = n$ , the operator  $T_J$  is hypercyclic, by hypothesis. Thus by Lemma 6.7,  $T_J$  satisfies the strong hypercyclicity criterion, and so there are dense sets  $D_{1,J}, D_{2,J} \subseteq X_J$ , a sequence  $\{n_{k,J}\}_{k=1}^\infty$ , and maps  $S_{n_{k,J}} : D_{2,J} \rightarrow X_J$  satisfying the conditions of the strong hypercyclicity criterion. We will not need the  $D_{1,J}$  sets as we already have  $D_1$  defined as needed.

Define  $\hat{D}_{2,J} \subseteq X$  so that  $\pi_J(\hat{D}_{2,J}) = D_{2,J}$  and  $\pi_{J^C}(\hat{D}_{2,J}) = \{0\}$  where  $J^C = \{1, \dots, p\} \setminus J$ . Thus  $\hat{D}_{2,J}$  is the set in  $X$  whose  $J$ -coordinates belong to  $D_{2,J}$  and whose non- $J$ -coordinates are zero. Now define

$$D_2 = \bigcup \{\hat{D}_{2,J} : J \subseteq \{1, \dots, p\} \text{ with } |J| = n\}.$$

Notice that the closure of  $D_2$  is the set  $\mathcal{K}_n$  described in Proposition 6.8, and hence by Proposition 6.8,  $D_2$  is  $n$ -weakly dense in  $X$ .

Now let us determine the sequence  $\{n_k\}$  and the approximate right inverses. Consider all the sequences  $\{\{n_{k,J}\}_{k=1}^\infty : J \subseteq \{1, 2, \dots, p\}, |J| = n\}$ , by possibly considering subsequences of

each of these, we may assume that they are pairwise disjoint. Then let  $\{n_k\}_{k=1}^\infty$  be an increasing enumeration of

$$\bigcup \{ \{n_{k,J}\}_{k=1}^\infty : J \subseteq \{1, 2, \dots, p\}, |J| = n \}.$$

Now, define  $S_{n_k}$ . Since  $n_k \in \bigcup \{ \{n_{k,J}\}_{k=1}^\infty : J \subseteq \{1, 2, \dots, p\}, |J| = n \}$ , there exist a unique  $i \geq 1$  and  $J$  with  $|J| = n$  such that  $n_k = n_{i,J}$ . In this case, we use the map  $S_{n_{i,J}} : D_{2,J} \subseteq X_J \rightarrow X_J$  to define  $S_{n_k} : D_2 \subseteq X \rightarrow X$  as

$$S_{n_k} y = (\pi_J | \hat{X}_J)^{-1} \circ S_{n_{i,J}} (\pi_J(y)) \quad \text{for } y \in \hat{D}_{2,J} \quad \text{and} \quad S_{n_k} y = 0 \quad \text{if } y \notin \hat{D}_{2,J}.$$

In other words, apply  $S_{n_{i,J}}$  to the  $J$ -coordinates of  $y$  and keep the other coordinates zero.

We can now check the conditions of Theorem 6.3. Clearly condition (1) of that theorem is satisfied, since in fact,  $T^n x \rightarrow 0$  as  $n \rightarrow \infty$  for each  $x \in D_1$ . For the second condition, if  $y \in D_2$ , then  $y \in \hat{D}_{2,J}$  for some  $J \subseteq \{1, \dots, p\}$  with  $|J| = n$ . Then the sequence  $\{n_{k,J}\}_{k=1}^\infty$  is a subsequence of  $\{n_k\}$  and since  $S_{n_{k,J}}$  is an approximate right inverse for  $T_J$  we have that  $T^{n_{k,J}} S_{n_{k,J}} y \rightarrow y$  as  $n_{k,J} \rightarrow \infty$  and  $S_{n_{k,J}} y \rightarrow 0$  as  $n_{k,J} \rightarrow \infty$ .

It follows that the conditions of Theorem 6.3 are satisfied, thus, there exists a dense  $G_\delta$  set  $\Omega \subseteq X$  such that for each  $x \in \Omega$ ,  $D_2 \subseteq \text{cl}[\text{Orb}(x, T)]$ . Since  $D_2$  is  $n$ -weakly dense in  $X$ , it follows that  $\text{Orb}(x, T)$  is also  $n$ -weakly dense in  $X$ , and thus  $T$  is  $n$ -weakly hypercyclic on  $X$ .  $\square$

**Remark.** Notice that the proof gives a stronger result than what the theorem actually states. In particular, the operators  $T_j$  need not have a dense set of vectors whose full orbits tend to zero; it suffices to have a sequence  $\{n_k\}$  with the following property: The direct sum of any  $n$  of the operators  $\{T_j\}_{j=1}^p$  satisfies the hypercyclicity criterion with respect to some subsequence of  $\{n_k\}$  and there are dense sets  $D_j \subseteq X_j$  such that  $T_j^{n_k} x \rightarrow 0$  as  $k \rightarrow \infty$  for each  $x \in D_j$ . Given this, the above proof shows that  $T = \bigoplus_{j=1}^p T_j$  is  $n$ -weakly hypercyclic.

Using a similar argument as that above and by applying Corollary 6.9 we get the following result as well.

**Theorem 6.11.** Suppose that  $T_j \in \mathcal{B}(X_j)$ ,  $1 \leq j < \infty$ , are a uniformly bounded sequence of bounded linear operators on separable Banach spaces each satisfying the strong hypercyclicity criterion. Suppose that  $1 \leq n < \infty$  and that the direct sum of any  $n$  of the operators  $\{T_j\}_{j=1}^\infty$  is hypercyclic, then the direct sum  $T = \bigoplus_{j=1}^\infty T_j$  is  $n$ -weakly hypercyclic.

The following result shows that weak forms of hypercyclicity imply hypercyclicity for direct sums of unilateral backward weighted shifts. Recall the definition of a jointly bounded orbit given in Definition 3.5.

**Proposition 6.12.** Suppose that  $B_1, \dots, B_n$  are unilateral backward weighted shifts on  $\ell^2(\mathbb{N})$  and let  $T = \bigoplus_{k=1}^n B_k$  which acts on  $(\ell^2(\mathbb{N}))^n = \bigoplus_{k=1}^n \ell^2(\mathbb{N})$ . Then the following are equivalent:

- (1)  $T$  is  $n$ -weakly hypercyclic.
- (2)  $T^*$  does not have a set  $S$  of non-zero vectors with a jointly bounded orbit where  $|S| \leq n$ .
- (3)  $T$  is hypercyclic.



**Proof.** By Proposition 3.6 (1) implies (2). Also, clearly (3) implies (1), thus it suffices to show that (2) implies (3). So, assume that (2) holds.

Suppose that the weighted shift  $B_j$  has weight sequence  $w_j = \{w_{j,i}\}_{i=0}^\infty$ , so that  $B_j(e_n) = w_{j,n}e_{n-1}$  for all  $n \geq 1$  and  $B_j(e_0) = 0$ . For each  $k \in \{1, \dots, n\}$ , let  $f_k = (0, 0, \dots, 0, e_0, 0, \dots, 0)$  where  $e_0$  is in the  $k$ th coordinate and where  $e_0 = (1, 0, 0, 0, \dots)$ . Then by (2) the set  $S = \{f_1, \dots, f_n\}$  cannot have a jointly bounded orbit under  $T^*$ . Thus it follows that

$$\sup_{m \geq 0} [\min\{\|T^{*m} f_1\|, \|T^{*m} f_2\|, \dots, \|T^{*m} f_n\|\}] = \infty.$$

Notice that

$$\|T^{*m} f_k\| = \|B_k^{*m} e_0\| = \left\| \left( \prod_{i=0}^{m-1} w_{k,i} \right) e_m \right\| = \prod_{i=0}^{m-1} w_{k,i}.$$

It follows that there is a sequence  $m_j \rightarrow \infty$  such that

$$\prod_{i=0}^{m_j-1} w_{k,i} \rightarrow \infty \quad \text{for all } 1 \leq k \leq n. \quad (*)$$

However, this is precisely the condition established by Salas [22] for  $T$  to be hypercyclic. Thus the result follows.  $\square$

We are now prepared to give a nice class of examples of operators that are  $n$ -weakly hypercyclic, but not  $(n+1)$ -weakly hypercyclic.

**Theorem 6.13.** Suppose that  $\{B_1, \dots, B_p\}$  are unilateral backward weighted shifts and  $T = \bigoplus_{k=1}^p B_k$ . For  $1 \leq n \leq p$ , we have that  $T$  is  $n$ -weakly hypercyclic if and only if the direct sum of any  $n$  of the operators  $\{B_1, \dots, B_p\}$  is hypercyclic.

**Proof.** Since a hypercyclic unilateral backward weighted shift satisfies the strong hypercyclicity criterion, if the direct sum of any  $n$  of the operators  $\{B_k\}_{k=1}^p$  is hypercyclic, then Theorem 6.10 implies that  $T$  is  $n$ -weakly hypercyclic. Conversely, if  $T$  is  $n$ -weakly hypercyclic, then it follows immediately that the direct sum of any  $n$  of the shifts will be  $n$ -weakly hypercyclic, and hence hypercyclic by Proposition 6.12.  $\square$

**Corollary 6.14.** For every positive integer  $n$ , there exist Hilbert space operators that are  $n$ -weakly hypercyclic but not  $(n+1)$ -weakly hypercyclic.

**Proof.** If  $B_1, B_2, \dots, B_p$  is a finite number of unilateral backward weighted shifts where  $B_k$  has weight sequence  $\{w_{k,i}\}_{i=0}^\infty$ , then  $\bigoplus_{k=1}^p B_k$  is hypercyclic if and only if there is a sequence of positive integers  $m_j \rightarrow \infty$  such that  $P_{k,m_j} := \prod_{i=0}^{m_j-1} w_{k,i} \rightarrow \infty$  as  $j \rightarrow \infty$  for all  $1 \leq k \leq n$ . Now simply choose  $w_{k,i} \in \{1, 2, \frac{1}{2}\}$  such that for each  $m \geq 0$ ,  $P_{k,m} = 1$  for all but at most  $n$  values of  $k$  and that for each finite set  $\mathcal{I} \subseteq \{1, \dots, p\}$  with  $|\mathcal{I}| = n$ , there is a sequence  $m_j \rightarrow \infty$  such that  $P_{k,m_j} \rightarrow \infty$  as  $j \rightarrow \infty$  for all  $k \in \mathcal{I}$ . So, if we choose  $p = n+1$ , and the  $\{w_{k,i}\}$  as described above, then the direct sum of any  $n$  of the operators  $\{B_1, \dots, B_n, B_{n+1}\}$  is hypercyclic, but the

direct sum of all  $(n + 1)$  of the operators is not hypercyclic. Thus  $T = \bigoplus_{k=1}^{n+1} B_k$  is  $n$ -weakly hypercyclic by Theorem 6.13, but by Proposition 6.12,  $T$  is not  $(n + 1)$ -weakly hypercyclic.  $\square$

The following special case of Theorem 6.13 is simple and nice with its two exclusive possibilities.

**Corollary 6.15.** *If  $B_1$  and  $B_2$  are any two hypercyclic backward unilateral weighted shifts, then exactly one of the following conditions holds:*

- (1)  $B_1 \oplus B_2$  is hypercyclic; or
- (2)  $B_1 \oplus B_2$  is 1-weakly hypercyclic but not 2-weakly hypercyclic.

The following result follows as above only uses Theorem 6.11.

**Theorem 6.16.** *Suppose that  $\{B_k\}_{k=1}^\infty$  is a uniformly bounded sequence of unilateral backward weighted shifts and  $T = \bigoplus_{k=1}^\infty B_k$ . For  $1 \leq n < \infty$ , we have that  $T$  is  $n$ -weakly hypercyclic if and only if the direct sum of any  $n$  of the operators  $\{B_k\}_{k=1}^\infty$  is hypercyclic.*

## 7. $n$ -Weakly supercyclic operators: basic properties

**Proposition 7.1.** *Suppose that  $T$  is a continuous linear operator on a locally convex space  $X$ , then a vector  $x \in X$  is a 1-weakly supercyclic vector for  $T$  if and only if  $x$  is a cyclic vector for  $T$ . In particular,  $T$  is 1-weakly supercyclic if and only if  $T$  is cyclic.*

**Proof.** If  $x \in X$ , then  $x$  is a 1-weakly supercyclic vector for  $T$  if and only if  $\mathbb{F} \cdot \text{Orb}(x, T)$  is 1-weakly dense in  $X$ . Furthermore,  $x$  is a cyclic vector for  $T$  if and only if  $\mathbb{F} \cdot \text{Orb}(x, T)$  has dense linear span in  $X$ . However, since  $\mathbb{F} \cdot \text{Orb}(x, T)$  is closed under scalar multiplication, Proposition 2.8 implies that  $\mathbb{F} \cdot \text{Orb}(x, T)$  is 1-weakly dense if and only if it has dense linear span. Thus the result follows.  $\square$

**Proposition 7.2.** *Suppose that  $T$  is a continuous linear operator on a locally convex space. If  $T$  is 2-weakly supercyclic, then*

- (1)  $T^*$  cannot have two linearly independent eigenvectors;
- (2)  $T$  has dense range.

**Proof.** (1) Suppose, by way of contradiction, that  $f_1$  and  $f_2$  are two linearly independent eigenvectors for  $T^*$  with eigenvalues  $\lambda_1$  and  $\lambda_2$  and that  $T$  is 2-weakly supercyclic with the vector  $x$  as a 2-weakly supercyclic vector for  $T$ . Then we have the following:

$$\begin{bmatrix} c\langle T^n x, f_1 \rangle \\ c\langle T^n x, f_2 \rangle \end{bmatrix} = \begin{bmatrix} c\langle x, T^{*n} f_1 \rangle \\ c\langle x, T^{*n} f_2 \rangle \end{bmatrix} = \begin{bmatrix} c\langle x, \lambda_1^n f_1 \rangle \\ c\langle x, \lambda_2^n f_2 \rangle \end{bmatrix} = c \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}^n \begin{bmatrix} \langle x, f_1 \rangle \\ \langle x, f_2 \rangle \end{bmatrix} = \begin{bmatrix} c\lambda_1^n \langle x, f_1 \rangle \\ c\lambda_2^n \langle x, f_2 \rangle \end{bmatrix}.$$

Since  $x$  is a 2-weakly supercyclic vector for  $T$  and since  $f_1$  and  $f_2$  are independent, the vectors on the left hand side of the equation above must be dense in  $\mathbb{F}^2$  as  $c \in \mathbb{F}$  and  $n \geq 0$  vary. Thus it follows that  $\left\{ \begin{bmatrix} c\lambda_1^n \langle x, f_1 \rangle \\ c\lambda_2^n \langle x, f_2 \rangle \end{bmatrix} : c \in \mathbb{F}, n \geq 0 \right\}$  is dense in  $\mathbb{F}^2$ . Thus quotients of the coordinates of these vectors must be dense in  $\mathbb{F}$ . So we have that

$$\left\{ \frac{\lambda_2^n \langle x, f_2 \rangle}{\lambda_1^n \langle x, f_1 \rangle} : n \geq 0 \right\}$$

must be dense in  $\mathbb{F}$ , but clearly it is not since it has the form  $\{a^n b : n \geq 0\}$  and such sets are not dense in  $\mathbb{F}$ .

(2) Let  $M$  be equal to the closure of the range of  $T$  and suppose that  $M \neq X$ . If  $x_0$  is a 2-weakly supercyclic vector for  $T$ , then  $T^n x_0 \in \text{Range}(T) \subseteq M$  for all  $n \geq 1$ , so  $\text{Orb}(x_0, T) \setminus \{x_0\} \subseteq M$ . Since  $x_0$  is a cyclic vector for  $T$ , by Proposition 7.1,  $\text{Orb}(x_0, T)$  has dense linear span in  $X$  and since  $M \neq X$ , we must have that  $x_0 \notin M$ . By the Hahn–Banach Theorem, there is an  $f \in X^*$  such that  $f(x_0) \neq 0$  and  $f(M) = \{0\}$ . Similarly, there is a  $g \in X^*$  such that  $g(x_0) = 0$  but  $g \neq 0$ . Then the map  $F : X \rightarrow \mathbb{F}^2$  given by  $F(x) = (f(x), g(x))$  is an onto (since  $f$  and  $g$  are linearly independent) continuous linear map, however  $F(\mathbb{F} \cdot \text{Orb}(x_0, T))$  is not dense in  $\mathbb{F}^2$  since  $F(T^n x_0)$  is zero in at least one of its coordinates for every  $n \geq 0$ . It follows that  $x_0$  is not a 2-weakly supercyclic vector, a contradiction. Hence  $T$  has dense range.  $\square$

**Corollary 7.3.** *If  $S$  is a pure subnormal operator, then  $S$  cannot be 2-weakly supercyclic.*

**Proof.** Suppose that  $S$  is a pure subnormal operator that is 2-weakly supercyclic, then  $S$  is 1-weakly supercyclic, and so by Proposition 7.1,  $S$  must be cyclic. However, according to Thomson’s Theorem (see [26] or [12]) the adjoint of every pure cyclic subnormal operator has a non-trivial open set of eigenvalues. Thus  $S^*$  has more than two independent eigenvectors contradicting Proposition 7.2.  $\square$

Also see Corollary 7.15 for another proof of the previous result that does not rely on the deep theorem of Thomson.

**Theorem 7.4.** *If  $n \geq 1$  and  $T$  is an  $n$ -weakly supercyclic operator on a locally convex space, then  $T^k$  is cyclic for all  $1 \leq k \leq n$ . In fact, if  $x$  is an  $n$ -weakly supercyclic vector for  $T$ , then  $x$  is also a cyclic vector for  $T^k$  for all  $1 \leq k \leq n$ .*

**Proof.** Fix a  $k$  satisfying  $1 \leq k \leq n$  and let  $x$  be an  $n$ -weakly supercyclic vector for  $T$ . By way of contradiction, suppose that  $x$  is not a cyclic vector for  $T^k$ . Then for each  $j$  satisfying  $0 \leq j \leq k$  we have that  $T^j x$  is not a cyclic vector for  $T^k$ . Thus the linear span of  $\{(T^k)^p (T^j x) : p \geq 0\}$  is not dense, so by the Hahn–Banach Theorem there exists a non-zero continuous linear functional  $f_j$  such that

$$\langle (T^k)^p (T^j x), f_j \rangle = 0 \quad \text{for all integers } p \geq 0.$$

Thus

$$\langle T^{kp+j} x, f_j \rangle = 0 \quad \text{for all } p \geq 0 \text{ and } 0 \leq j \leq k. \quad (*)$$

If we set  $F = \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix}$ , then  $F : X \rightarrow \mathbb{F}^k$  is linear and non-zero, but we will show that  $F(\mathbb{F} \cdot \text{Orb}(x, T))$  is not dense in  $F(X)$ . To see this, notice that by Eq. (\*), for every  $m \geq 0$  and for any  $c \in \mathbb{F}$ , at least one coordinate of  $F(cT^m x)$  will be equal to zero. On the other hand, since each  $f_j$  is non-zero, then the kernel of  $f_j$ ,  $\ker(f_j)$  is a closed nowhere dense subspace of  $X$ . Since a

finite union of nowhere dense sets is still nowhere dense, we must have that  $X \neq \bigcup_{j=1}^k \ker(f_j)$ . It follows that the range of  $F$ ,  $F(X)$ , will contain vectors in which all coordinates are non-zero. Thus,  $F(\mathbb{F} \cdot \text{Orb}(x, T))$  is not dense in  $F(X)$ . So  $T$  is not  $n$ -weakly supercyclic, a contradiction. Therefore we must have that  $x$  is a cyclic vector for  $T^k$ .  $\square$

In [25], Shkarin proved the surprising result that a bilateral weighted shift operator  $T$  on  $\ell^2(\mathbb{Z})$  is supercyclic if and only if  $T^2$  is cyclic! Using this fact together with Theorem 7.4 we get the following result.

**Corollary 7.5.** *If  $T$  is a bilateral weighted shift operator on  $\ell^2(\mathbb{Z})$  and if  $T$  is 2-weakly supercyclic, then  $T$  is supercyclic.*

It is known that if  $T$  is an operator on a space  $X$  and  $I$  denotes the identity operator on the scalar field  $\mathbb{F}$ , then  $T \oplus I$  is supercyclic on  $X \oplus \mathbb{F}$  if and only if  $T$  is hypercyclic. The next result gives an  $n$ -weak version of this fact.

**Theorem 7.6.** *If  $T$  is a continuous linear operator on a locally convex space  $X$  over  $\mathbb{F}$ ,  $I$  denotes the identity operator on  $\mathbb{F}$ , and  $n \geq 1$ , then each of the following statements implies the next one:*

- (1)  $T \oplus I$  is  $(n+1)$ -weakly supercyclic on  $X \oplus \mathbb{F}$ .
- (2)  $T$  has a vector that is both an  $n$ -weakly hypercyclic vector and an  $(n+1)$ -weakly supercyclic vector.
- (3)  $T$  is  $n$ -weakly hypercyclic on  $X$ .
- (4)  $T \oplus I$  is  $n$ -weakly supercyclic on  $X \oplus \mathbb{F}$ .

The following implications hold: (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4).

**Proof.** (1)  $\Rightarrow$  (2). Suppose that  $T \oplus I$  is  $(n+1)$ -weakly supercyclic on  $X \oplus \mathbb{F}$  and let  $(v, \alpha) \in X \oplus \mathbb{F}$  be an  $(n+1)$ -weakly supercyclic vector for  $T \oplus I$ . Then we claim that  $v$  is both an  $n$ -weakly hypercyclic vector for  $T$  and an  $(n+1)$ -weakly supercyclic vector for  $T$ . It follows by Proposition 3.2 that since  $(v, \alpha)$  is an  $(n+1)$ -weakly supercyclic vector for  $T \oplus I$ , then  $v$  must be an  $(n+1)$ -weakly supercyclic vector for  $T$ . So, it remains to show that  $v$  is also an  $n$ -weakly hypercyclic for  $T$ . That is,  $\text{Orb}(v, T)$  is  $n$ -weakly dense in  $X$ . So, let  $F : X \rightarrow \mathbb{F}^n$  be an onto continuous linear map and we must show that  $F(\text{Orb}(v, T))$  is dense in  $\mathbb{F}^n$ . Let  $w \in \mathbb{F}^n$  and we will show that  $w$  belongs to the closure of  $F(\text{Orb}(v, T))$ . To see this, define  $G : X \oplus \mathbb{F} \rightarrow \mathbb{F}^n \times \mathbb{F} = \mathbb{F}^{n+1}$  by  $G(x, y) = (F(x), y)$ . Then  $G : X \oplus \mathbb{F} \rightarrow \mathbb{F}^{n+1}$  is an onto continuous linear map. Since  $(v, \alpha)$  is an  $(n+1)$ -weakly supercyclic vector for  $T \oplus I$ , then  $G(\mathbb{F} \cdot \text{Orb}((v, \alpha), T \oplus I))$  is dense in  $\mathbb{F}^{n+1}$ , so there exist a sequence of scalars  $\{c_k\}_{k=1}^\infty$  and a sequence of positive integers  $\{n_k\}_{k=1}^\infty$  such that  $G(c_k(T \oplus I)^{n_k}(v, \alpha)) \rightarrow (w, \alpha)$  as  $k \rightarrow \infty$ . Thus,  $F(c_k T^{n_k} v) \rightarrow w$  and  $c_k \alpha \rightarrow \alpha$ . Since  $(v, \alpha)$  is an  $(n+1)$ -supercyclic vector for  $T \oplus I$ , then  $\alpha$  must be non-zero, thus  $c_k \rightarrow 1$ , which implies that  $F(T^{n_k} v) \rightarrow w$ . Thus  $w$  is in the closure of  $F(\text{Orb}(v, T))$ , as desired. Since  $w \in \mathbb{F}^n$  was arbitrary, then  $F(\text{Orb}(v, T))$  is dense in  $\mathbb{F}^n$ , and thus  $v$  is an  $n$ -weakly hypercyclic vector for  $T$ .

(2)  $\Rightarrow$  (3). This is obvious.

(3)  $\Rightarrow$  (4). Suppose that  $v \in X$  is an  $n$ -weakly hypercyclic vector for  $T$  and we will show that  $(v, 1) \in X \oplus \mathbb{F}$  is an  $n$ -weakly supercyclic vector for  $T \oplus I$ . Let  $F : X \oplus \mathbb{F} \rightarrow \mathbb{F}^n$  be an onto

continuous linear map and we will show that  $F(\mathbb{F} \cdot \text{Orb}((v, 1), T \oplus I))$  is dense in  $\mathbb{F}^n$ ; we will consider two cases.

First, define the map  $\hat{F} : X \rightarrow \mathbb{F}^n$  as follows:  $\hat{F}(x) = F(x, 0)$  for  $x \in X$ .

**Case 1.**  $F(0, 1) = 0$ .

Since  $F : X \oplus \mathbb{F} \rightarrow \mathbb{F}^n$  is onto and since we are assuming that  $F(0, 1) = 0$ , then it follows that  $\hat{F} : X \rightarrow \mathbb{F}^n$  is onto. Since  $v$  is an  $n$ -weakly hypercyclic vector for  $T$  and  $\hat{F}$  is an onto, then it follows that

$$\hat{F}(\text{Orb}(v, T)) \text{ is dense in } \mathbb{F}^n. \quad (*)$$

Since  $F(T^n v, 1) = F(T^n v, 0) + F(0, 1) = F(T^n v, 0) + 0 = \hat{F}(T^n v)$ , it follows easily from  $(*)$  above that  $F(\text{Orb}((v, 1), T \oplus I))$  is dense in  $\mathbb{F}^n$ . Thus this case is established.

**Case 2.**  $F(0, 1) \neq 0$ .

In this case we may suppose that  $F(0, 1) = e_n = (0, 0, \dots, 0, 1)$ , otherwise, since  $F(0, 1) \neq 0$ , we may choose an invertible matrix  $M : \mathbb{F}^n \rightarrow \mathbb{F}^n$  such that  $MF(0, 1) = e_n$  and then consider the map  $MF$  instead of  $F$ . Thus we will assume that  $F(0, 1) = e_n$ . We must show that  $F(\mathbb{F} \cdot \text{Orb}((v, 1), T \oplus I))$  is dense in  $\mathbb{F}^n$ .

Let  $w = (w_1, w_2, \dots, w_n) \in \mathbb{F}^n$  and assume that  $w_n \neq 0$ ; such vectors are dense in  $\mathbb{F}^n$ . We will show that  $w$  is in the closure of  $F(\mathbb{F} \cdot \text{Orb}((v, 1), T \oplus I))$ , thus showing that  $F(\mathbb{F} \cdot \text{Orb}((v, 1), T \oplus I))$  is dense in  $\mathbb{F}^n$ .

Let  $\hat{w} = \frac{1}{w_n}(w_1, w_2, \dots, w_{n-1}, 0)$  and consider  $\hat{F} : X \rightarrow \mathbb{F}^n$  as defined above. Since  $\hat{F}$  is an onto continuous linear map and  $\hat{w} \in \mathbb{F}^n$ , and since  $v$  is an  $n$ -weakly hypercyclic vector for  $T$ , then there exists a sequence of positive integers  $\{n_k\}_{k=1}^\infty$  such that  $\hat{F}(T^{n_k} v) \rightarrow \hat{w} = \frac{1}{w_n}(w_1, w_2, \dots, w_{n-1}, 0)$  as  $k \rightarrow \infty$ .

Thus

$$\begin{aligned} F((T \oplus I)^{n_k}(v, 1)) &= F(T^{n_k} v, 1) = F(T^{n_k} v, 0) + F(0, 1) = \hat{F}(T^{n_k} v) + e_n \\ &\rightarrow \frac{1}{w_n}(w_1, w_2, \dots, w_{n-1}, 0) + (0, \dots, 0, 1) = \frac{1}{w_n}(w_1, w_2, \dots, w_n). \end{aligned}$$

Thus,  $F(w_n(T \oplus I)^{n_k}(v, 1)) \rightarrow (w_1, \dots, w_n)$ . Thus,  $F(\mathbb{F} \cdot \text{Orb}((v, 1), T \oplus I))$  is dense in  $\mathbb{F}^n$ . This case is now established.

It now follows that  $(v, 1)$  is an  $n$ -weakly supercyclic vector for  $T$ . Thus we have established that (3) implies (4).  $\square$

**Corollary 7.7.** *For every  $n \geq 1$ , there exists a bounded linear operator  $T$  on a Hilbert space that is  $n$ -weakly supercyclic but not  $(n + 2)$ -weakly supercyclic.*

**Proof.** If  $n = 1$ , then the bilateral (unweighted) shift  $B$  on  $\ell^2(\mathbb{Z})$  is a 1-weakly supercyclic operator (= a cyclic operator) that is not 2-weakly supercyclic. For if  $B$  was 2-weakly supercyclic, then by Theorem 7.4,  $B^2$  would be cyclic, which it is not.

Now suppose that  $n \geq 2$ . By Corollary 6.14, there exists an operator  $T$ , which is a direct sum of backward unilateral weighted shifts on  $\ell^2(\mathbb{N})$ , such that  $T$  is  $n$ -weakly hypercyclic, but not  $(n+1)$ -weakly hypercyclic. Thus, by Theorem 7.6, if  $I$  is the identity operator on a one-dimensional space, then  $T \oplus I$  is  $n$ -weakly supercyclic. However, since  $T$  is not  $(n+1)$ -weakly hypercyclic, then again by Theorem 7.6,  $T \oplus I$  cannot be  $(n+2)$ -weakly supercyclic.  $\square$

The previous example begs the question of whether there are operators that are  $n$ -weakly supercyclic but not  $(n+1)$ -weakly supercyclic. Perhaps a necessary and sufficient version of Theorem 7.6 would do the trick. However, the previous example does show that non-trivial  $n$ -weakly supercyclic operators do exist.

Shkarin [24] proved a weak angle criterion to show that a vector with certain properties is not a weakly supercyclic vector (also see [5, p. 240]). We will show that Shkarin's weak angle criterion actually guarantees that the vector is not a 2-weakly supercyclic vector (see Theorem 7.10). Montes-Rodriguez and Shkarin [19] also proved a result similar to Theorem 7.8 below, however their condition required an auxiliary Hilbert–Schmidt operator and relied on facts about Gaussian measures in Hilbert spaces. Our result relies on Ball's Theorem (see Theorem 3.1).

**Theorem 7.8** (*n-Weak angle criterion*). *Suppose that  $T$  is a bounded linear operator on a Banach space  $X$  and that  $x \in X$ . If there is a set of  $n$  non-zero continuous linear functionals  $\{f_1, \dots, f_n\}$  on  $X$  such that*

$$\sum_{k=0}^{\infty} \left( \frac{\min\{|f_1(T^k x)|, |f_2(T^k x)|, \dots, |f_n(T^k x)|\}}{\|T^k x\|} \right)^p < \infty \quad (3)$$

*then  $x$  is not a  $(2n)$ -weakly supercyclic vector for  $T$  provided that either one of the following holds: (i)  $p = 1$ , or (ii)  $p = 2$  and  $X$  is a Hilbert space.*

The following proposition is needed for the proof of the previous theorem. Its statement and proof are generalized and adapted from that found in [24] or [5, p. 241] to the  $n$ -weak setting.

In what follows, for a Banach space  $(X, \|\cdot\|)$ ,  $X^n$  will denote a Banach space that is the direct sum of  $n$  copies of  $X$  endowed with a norm where each of the coordinate projection maps is continuous. It is easy to check that any two such norms on  $X^n$  are equivalent. In fact, let  $X_{\ell^\infty}^n$  denote the Banach space obtained by endowing  $X^n$  with the  $\ell^\infty$  norm as follows: If  $x = (x_1, \dots, x_n) \in X^n$ , then  $\|x\|_\infty = \max\{\|x_1\|, \dots, \|x_n\|\}$ . Then if  $\|\cdot\|_*$  is any norm on  $X^n$  such that the coordinate projection maps are continuous, then the identity map  $i : (X^n, \|\cdot\|_*) \rightarrow X_{\ell^\infty}^n$  is continuous. Since  $i$  is clearly a bijection, then the Open Mapping Theorem implies that  $i$  is an isomorphism, hence the two norms are equivalent. We will let  $X_{\ell^2}^n$  denote the  $\ell^2$ -direct sum of  $n$  copies of  $X$ , so if  $x = (x_1, \dots, x_n) \in X^n$ , then  $\|x\|_{X_{\ell^2}^n}^2 = \|x\|_2^2 = \sum_{k=1}^n \|x_k\|^2$ . If  $X$  is a Hilbert space, then  $X_{\ell^2}^n$  is also a Hilbert space.

**Proposition 7.9.** *Suppose that  $\{x_k\}_{k=1}^\infty$  is a sequence in a Banach space  $X$  over  $\mathbb{F}$ ,  $z \in X$ , and  $n \in \mathbb{N}$ . If  $z$  belongs to the  $2n$ -weak closure of  $\mathbb{F} \cdot \{x_k : k \geq 1\}$  and  $\{f_1, f_2, \dots, f_n\} \subseteq X^*$  satisfy  $f_k(z) \neq 0$  for all  $1 \leq k \leq n$ , then*

$$\left( \frac{z}{f_1(z)}, \frac{z}{f_2(z)}, \dots, \frac{z}{f_n(z)} \right) \in X^n$$

belongs to the 1-weak closure of

$$E = \left\{ \left( \frac{x_k}{f_1(x_k)}, \frac{x_k}{f_2(x_k)}, \dots, \frac{x_k}{f_n(x_k)} \right) : k \geq 1 \text{ \& } f_j(x_k) \neq 0 \forall 1 \leq j \leq n \right\}$$

in  $X^n$ .

**Proof.** Let  $z^{(n)} = (\frac{z}{f_1(z)}, \frac{z}{f_2(z)}, \dots, \frac{z}{f_n(z)})$ . Let  $g \in (X^n)^*$ , then by Proposition 2.6, in order to show that  $z^{(n)}$  belongs to the 1-weak closure of  $E$  we must show that  $g(z^{(n)})$  belongs to the closure of  $g(E)$ . Since  $g \in (X^n)^*$ , then there exists  $\{g_1, \dots, g_n\} \subseteq X^*$  such that for any  $v = (v_1, v_2, \dots, v_n) \in X^n$ ,  $g(v) = \sum_{j=1}^n g_j(v_j)$ .

Define  $F : X \rightarrow \mathbb{F}^{2n}$  by  $F(x) = (f_1(x), \dots, f_n(x), g_1(x), \dots, g_n(x))$ . Since  $z$  belongs to the  $2n$ -weak closure of  $\mathbb{F} \cdot \{x_k : k \geq 1\}$ , then there exist  $c_k \in \mathbb{F}$  and  $n_k \in \mathbb{N}$  such that  $F(c_k x_{n_k}) \rightarrow F(z)$  as  $k \rightarrow \infty$ . Thus we have

$$F(c_k x_{n_k}) = \begin{bmatrix} f_1(c_k x_{n_k}) \\ \vdots \\ f_n(c_k x_{n_k}) \\ g_1(c_k x_{n_k}) \\ \vdots \\ g_n(c_k x_{n_k}) \end{bmatrix} \rightarrow F(z) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_n(z) \\ g_1(z) \\ \vdots \\ g_n(z) \end{bmatrix} \text{ as } k \rightarrow \infty.$$

Since  $f_j(z) \neq 0$  for all  $j \in \{1, \dots, n\}$ , then  $f_j(c_k x_{n_k}) \neq 0$  for all  $j \in \{1, \dots, n\}$  and for all large  $k$ , thus we may divide by these quantities and obtain the following:

$$\begin{bmatrix} g_1(\frac{x_{n_k}}{f_1(x_{n_k})}) \\ \vdots \\ g_n(\frac{x_{n_k}}{f_n(x_{n_k})}) \end{bmatrix} = \begin{bmatrix} g_1(\frac{c_k x_{n_k}}{f_1(c_k x_{n_k})}) \\ \vdots \\ g_n(\frac{c_k x_{n_k}}{f_n(c_k x_{n_k})}) \end{bmatrix} = \begin{bmatrix} \frac{g_1(c_k x_{n_k})}{f_1(c_k x_{n_k})} \\ \vdots \\ \frac{g_n(c_k x_{n_k})}{f_n(c_k x_{n_k})} \end{bmatrix} \rightarrow \begin{bmatrix} g_1(\frac{z}{f_1(z)}) \\ \vdots \\ g_n(\frac{z}{f_n(z)}) \end{bmatrix}. \quad (*)$$

Now let

$$v_k = (v_{k,1}, v_{k,2}, \dots, v_{k,n}) = \left( \frac{x_{n_k}}{f_1(x_{n_k})}, \frac{x_{n_k}}{f_2(x_{n_k})}, \dots, \frac{x_{n_k}}{f_n(x_{n_k})} \right)$$

so that  $v_{k,j} = \frac{x_{n_k}}{f_j(x_{n_k})}$  for  $1 \leq j \leq n$ . Then  $v_k \in E$  and by looking at the two ends of equation (\*) above we see that

$$g(v_k) = \sum_{j=1}^n g_j(v_{k,j}) = \sum_{j=1}^n g_j\left(\frac{x_{n_k}}{f_j(x_{n_k})}\right) \rightarrow \sum_{j=1}^n g_j\left(\frac{z}{f_j(z)}\right) = g(z^{(n)}).$$

Thus,  $z^{(n)}$  belongs to the 1-weak closure of  $E$  in  $X^n$ .  $\square$

**Proof of Theorem 7.8.** By way of contradiction, assume that the vector  $x$ , in the statement of the theorem, is a  $2n$ -weakly supercyclic vector for  $T$ , then  $\mathbb{F} \cdot \text{Orb}(x, T)$  is  $2n$ -weakly dense in  $X$ . Note that our hypothesis implies that  $\dim(X) \geq 2$ .



Since for each  $j$  the functional  $f_j$  is non-zero we get that  $\bigcap_{j=1}^n \{v: f_j(v) \neq 0\} \neq \emptyset$ , otherwise we would have  $X = \bigcup_{j=1}^n \{v: f_j(v) = 0\}$  and then since  $X$  is represented as a finite union of closed sets, then one of the closed sets must have non-empty interior, implying that  $f_j$  equals zero for some  $j$ , contradicting our hypothesis. So,  $\bigcap_{j=1}^n \{v: f_j(v) \neq 0\} \neq \emptyset$ .

Since  $\bigcap_{j=1}^n \{v: f_j(v) \neq 0\}$  is a non-empty open set in  $X$ , it has dimension at least two, whereas  $\mathbb{F} \cdot \text{Orb}(x, T)$  is a countable union of one-dimensional subspaces, so the Baire Category Theorem implies that  $\bigcap_{j=1}^n \{v: f_j(v) \neq 0\}$  is not a subset of  $\mathbb{F} \cdot \text{Orb}(x, T)$ . So,  $[\bigcap_{j=1}^n \{v: f_j(v) \neq 0\}] \setminus (\mathbb{F} \cdot \text{Orb}(x, T)) \neq \emptyset$ . Let  $z \in [\bigcap_{j=1}^n \{v: f_j(v) \neq 0\}] \setminus (\mathbb{F} \cdot \text{Orb}(x, T))$ . Then  $f_j(z) \neq 0$  for all  $j \in \{1, \dots, n\}$  and  $z \notin \mathbb{F} \cdot \text{Orb}(x, T)$ .

Now since  $x$  is a  $2n$ -weakly supercyclic vector for  $T$ , then  $z$  belongs to the  $2n$ -weak closure of  $\mathbb{F} \cdot \{T^k x\}_{k=0}^\infty$ , thus by Proposition 7.9,

$$z^{(n)} := \left( \frac{z}{f_1(z)}, \frac{z}{f_2(z)}, \dots, \frac{z}{f_n(z)} \right) \in X_{\ell^2}^n$$

will belong to the 1-weak closure of

$$E = \left\{ \left( \frac{T^k x}{f_1(T^k x)}, \frac{T^k x}{f_2(T^k x)}, \dots, \frac{T^k x}{f_n(T^k x)} \right) : k \in J \right\} \quad \text{in } X_{\ell^2}^n$$

where  $J = \{k \in \mathbb{N} : f_j(T^k x) \neq 0 \forall 1 \leq j \leq n\}$ .

Recall that  $X_{\ell^p}^n$  denotes the  $\ell^p$ -direct sum of  $n$  copies of  $X$ . Notice that  $z^{(n)} \notin E$  since  $z \in X \setminus (\mathbb{F} \cdot \text{Orb}(x, T))$ . Since  $z^{(n)}$  belongs to the 1-weak closure of  $E$ , then  $E$  must not be 1-weakly closed. However, we will now show that  $E$  is 1-weakly closed, and thus obtain a contradiction. To see this we will simply apply Ball's Theorem (Theorem 3.1). Notice that if

$$v_k = \left( \frac{T^k x}{f_1(T^k x)}, \frac{T^k x}{f_2(T^k x)}, \dots, \frac{T^k x}{f_n(T^k x)} \right) \in E$$

then

$$\|v_k\|_2 \geq \|v_k\|_\infty = \max_{1 \leq j \leq n} \frac{\|T^k x\|}{|f_j(T^k x)|} = \frac{\|T^k x\|}{\min_{1 \leq j \leq n} |f_j(T^k x)|}.$$

Thus, letting  $p = 2$  if  $X$  is a Hilbert space and  $p = 1$  otherwise, then

$$\sum_{k \in J} \frac{1}{\|v_k\|_2^p} \leq \sum_{k \in J} \left( \frac{\min_{1 \leq j \leq n} |f_j(T^k x)|}{\|T^k x\|} \right)^p \leq \sum_{k=1}^\infty \left( \frac{\min_{1 \leq j \leq n} |f_j(T^k x)|}{\|T^k x\|} \right)^p < \infty.$$

Where the last sum above converges by hypothesis. Now since  $X_{\ell^2}^n$  is a Hilbert space whenever  $X$  is a Hilbert space and is otherwise a Banach space, then according to Ball's Theorem  $E = \{v_k : k \in J\}$  is 1-weakly closed in  $X_{\ell^2}^n$ . Thus we have a contradiction. So, it follows that  $x$  cannot be a  $2n$ -weakly supercyclic vector for  $T$ .  $\square$

The following 1-weak angle criterion was used to prove some of the results about the 2-weak supercyclicity of matrices stated in Section 4. It is an immediate corollary of the  $n$ -weak angle

criterion (Theorem 7.8). Shkarin [24] proved the very nice weak angle criterion stated below although he did not use the 2-weakly dense terminology, but concluded that  $x$  is not a weakly supercyclic vector (also see [5, p. 240]).

**Theorem 7.10** (1-Weak angle criterion). *Suppose that  $T$  is a bounded linear operator on a Banach space  $X$  and that  $x \in X$ . If there is a non-zero continuous linear functional  $f$  on  $X$  such that*

$$\sum_{n=0}^{\infty} \left( \frac{|f(T^n x)|}{\|T^n x\|} \right)^p < \infty$$

*then  $x$  is not a 2-weakly supercyclic vector for  $T$  provided that either one of the following holds: (i)  $p = 1$ , or (ii)  $p = 2$  and  $X$  is a Hilbert space.*

The following result is a nice easy consequence of the 1-weak angle criterion and is easy to apply in various situations.

**Corollary 7.11** (Weak ratio criterion). *If  $T$  is a bounded linear operator on a Banach space  $X$  and  $T$  has two invariant subspaces  $\mathcal{M}$  and  $\mathcal{N}$  that are complementary in  $X$  and such that for every  $x \in \mathcal{M}$  and for every non-zero  $y \in \mathcal{N}$  we have that*

$$\sum_{n=0}^{\infty} \left( \frac{\|T^n x\|}{\|T^n y\|} \right)^a < \infty$$

*then  $T$  is not 2-weakly supercyclic on  $X$ ; provided that  $a = 1$  or  $a = 2$  and  $X$  is a Hilbert space.*

**Proof.** Suppose, by way of contradiction, that  $T$  is 2-weakly supercyclic and that  $v \in X$  is a 2-weakly supercyclic vector for  $T$ . Also suppose that  $v = x + y$  where  $x \in \mathcal{M}$  and  $y \in \mathcal{N}$ . Then both  $x$  and  $y$  must be non-zero, otherwise  $v$  will belong to one of the two invariant subspaces, in which case  $v$  is not even a cyclic vector (= a 1-weakly supercyclic vector), let alone a 2-weakly supercyclic vector for  $T$  (see Theorem 7.4). Since  $T$  is 2-weakly supercyclic,  $T$  will have dense range (see Proposition 7.2), thus  $T^n v$  will also be a 2-weakly supercyclic vector for  $T$ , thus using similar reasoning as above we have that  $T^n x$  and  $T^n y$  are both non-zero for every  $n \geq 0$ . Next choose  $f \in X^* \setminus \{0\}$  such that  $f(\mathcal{N}) = \{0\}$ . Then we have that  $f(T^n v) = f(T^n x) + f(T^n y) = f(T^n x)$  since  $y \in \mathcal{N}$  and  $\mathcal{N}$  is invariant for  $T$ . Also there is a  $c > 0$  such that  $\|T^n v\| \geq c\|T^n y\|$ . To see this notice that if  $P : X \rightarrow \mathcal{N}$  is the idempotent mapping onto  $\mathcal{N}$  having  $\mathcal{M}$  as its kernel, then  $P(T^n v) = T^n y$  and thus  $\|T^n y\| = \|P(T^n v)\| \leq \|P\|\|T^n v\|$ , thus we may take  $c = 1/\|P\|$ . Thus we have

$$\sum_{n=0}^{\infty} \left( \frac{|f(T^n v)|}{\|T^n v\|} \right)^a = \sum_{n=0}^{\infty} \left( \frac{|f(T^n x)|}{\|T^n v\|} \right)^a \leq \sum_{n=0}^{\infty} \left( \frac{|f(T^n x)|}{c\|T^n y\|} \right)^a \leq \frac{\|f\|^a}{c^a} \sum_{n=0}^{\infty} \left( \frac{\|T^n x\|}{\|T^n y\|} \right)^a.$$

Since by assumption the sum on the right hand side converges, then so does the sum on the left hand side, thus by the weak angle criterion (Theorem 7.10),  $v$  is not a 2-weakly supercyclic vector for  $T$ , a contradiction. Thus  $T$  is not 2-weakly supercyclic.  $\square$

The weak ratio criterion was used to obtain some of the results in Section 4 to show that certain matrices are not 2-weakly supercyclic and will now be used to obtain a result that 2-weakly supercyclic operators have a “supercyclicity circle” just like supercyclic operators; see Feldman, Miller, and Miller [16] or [5, p. 12].

**Lemma 7.12.** *If  $T_1$  and  $T_2$  are two bounded linear operators on Banach spaces  $X_1$  and  $X_2$  respectively and there is an  $r > 0$  such that  $\sigma(T_1) \subseteq \{z \in \mathbb{C}: |z| < r\}$  and  $\sigma(T_2) \subseteq \{z \in \mathbb{C}: |z| > r\}$ , then  $T = T_1 \oplus T_2$  is not 2-weakly supercyclic.*

**Proof.** We will simply apply the weak ratio criterion with  $\mathcal{M} = X_1 \oplus \{0\}$  and  $\mathcal{N} = \{0\} \oplus X_2$ , which are complementary subspaces of  $X_1 \oplus X_2$  and invariant for  $T$ . Choose  $r_1, r_2$  such that  $0 < r_1 < r < r_2$  and such that  $\sigma(T_1) \subseteq \{z \in \mathbb{C}: |z| < r_1\}$  and  $\sigma(T_2) \subseteq \{z \in \mathbb{C}: |z| > r_2\}$ , then using standard estimates on the growth of orbits it follows that there are constants  $c_1, c_2 > 0$  such that  $\|T_1^n x\| \leq c_1 r_1^n \|x\|$  for all  $n \geq 0$  and all  $x \in X_1$ . Also,  $\|T_2^n y\| \geq c_2 r_2^n \|y\|$  for all  $n \geq 0$  and for all  $y \in X_2$ . Thus,

$$\sum_{n=0}^{\infty} \left( \frac{\|T^n(x \oplus 0)\|}{\|T^n(0 \oplus y)\|} \right) = \sum_{n=0}^{\infty} \left( \frac{\|T_1^n x\|}{\|T_2^n y\|} \right) \leq \sum_{n=0}^{\infty} \left( \frac{c_1 r_1^n \|x\|}{c_2 r_2^n \|y\|} \right) = C \sum_{n=0}^{\infty} \left( \frac{r_1}{r_2} \right)^n < \infty$$

since  $r_1 < r_2$ . Thus by the weak ratio criterion (Corollary 7.11) (with  $a = 1$ ) we have that  $T$  is not 2-weakly supercyclic.  $\square$

The proof of the following result is identical to the proof of Theorem 6.2 in [16] except that it makes use of Lemma 7.12 above instead of Lemmas 6.3 and 6.4 in [16].

**Theorem 7.13.** *Suppose that  $T$  is a bounded linear operator on a separable Banach space  $X$ . If  $T$  is 2-weakly supercyclic, then there is a (possibly degenerate) circle  $C_r = \{z \in \mathbb{C}: |z| = r\}$ ,  $r \geq 0$ , such that  $\sigma(T^*|_{\mathcal{M}}) \cap C_r \neq \emptyset$  for every weak\*-closed  $T^*$  invariant subspace  $\mathcal{M} \subseteq X^*$ . In particular, every component of  $\sigma(T)$  intersects the (possibly degenerate) circle  $C_r$ .*

One immediate consequence of the previous theorem is the following result.

**Example 7.14.** If a normal operator  $T$  on a Hilbert space is 2-weakly supercyclic, then  $T$  must be a multiple of a unitary operator.

It was shown by Bayart and Matheron [4], also see [5, p. 254], that the unitary operator  $N_\mu = M_z$  on  $L^2(\mu)$  is weakly supercyclic if  $\mu$  is supported on a Kronecker set (a thin type of Cantor set) on the unit circle. In the same paper, Bayart and Matheron also proved the very nice result that a weakly supercyclic hyponormal operator must be a multiple of a unitary. Below we see that this follows easily for subnormal operators since it is true for normal operators.

**Corollary 7.15.** *If  $S$  is a subnormal operator that is 2-weakly supercyclic, then  $S$  must be a multiple of a singular unitary operator.*

**Proof.** In [14] (also see [10, Theorems 4.3 and 4.4]) it is shown that if  $S$  is a subnormal operator on a Hilbert space  $\mathcal{H}$  with spectral measure  $\mu$ , then there exists a bounded linear map

$A : \mathcal{H} \rightarrow L^2(\mu)$  that has dense range and satisfies  $AS = N_\mu A$  where  $N_\mu$  is the normal operator of multiplication by  $z$  on  $L^2(\mu)$ . If  $S$  is 2-weakly supercyclic, then by Proposition 3.2,  $N$  is also 2-weakly supercyclic. However, then by Theorem 7.13 it follows that the spectrum of  $N$  must be contained in a circle. Hence it follows that  $N$  is a multiple of a unitary operator. It now follows easily from Theorem 8.1 that the unitary operator must be singular.  $\square$

Shkarin [24] proved that for  $1 \leq p \leq 2$  a weakly supercyclic bilateral weighted shift on  $\ell^p(\mathbb{Z})$  must be supercyclic. Using the 1-weak angle criterion (Theorem 7.10) and following Shkarin's proof in [24] or [5, p. 242] one can obtain the following result.

**Example 7.16.** If  $T$  is a 2-weakly supercyclic bilateral weighted shift on  $\ell^p(\mathbb{Z})$  for  $1 \leq p \leq 2$ , then  $T$  is supercyclic on  $\ell^p(\mathbb{Z})$ .

**Example 7.17.** The Volterra operator  $(Vf)(x) = \int_0^x f(t) dt$  on  $L^p[0, 1]$  is not 2-weakly supercyclic. The proof by Montes and Shkarin [19, Theorem 2.4, p. 43], which shows that  $V$  is not weakly supercyclic, actually shows that  $V$  is not 2-weakly supercyclic; since the open set  $U$  at the end of their proof is a 2-weakly open set.

**Example 7.18 (Linear fractional composition operators).** Following Montes and Shkarin in [19], define the weighted Dirichlet spaces  $S_\nu$  to be the set of all analytic functions  $f(z) = \sum_{n=0}^\infty a_n z^n$  on the unit disk such that  $\|f\|_{S_\nu}^2 = \sum_{n=0}^\infty (n+1)^{2\nu} |a_n|^2$  is finite. In [19] it is shown that if a linear fractional composition operator on  $S_\nu$  is weakly supercyclic, then it is supercyclic. In fact, using results of this paper, their method of proof actually shows that *if a linear fractional composition operator on  $S_\nu$  is 2-weakly supercyclic, then it is supercyclic*. To prove this one only needs to use Proposition 7.2 and Theorem 7.13 in this paper along with the arguments in [19].

## 8. Cohyponormal operators

In [24], Shkarin proves that there exists a continuous Borel measure  $\mu$  on the unit circle such that  $N_\mu = M_z$  on  $L^2(\mu)$  is weakly supercyclic, but the Fourier coefficients  $\hat{\mu}(n) = \int z^n d\mu \rightarrow 0$  as  $|n| \rightarrow \infty$ . This latter condition implies that every scaled orbit of  $N_\mu$  is weakly sequentially closed. Hence  $N_\mu$  is not weakly sequentially supercyclic. If we assume the stronger condition on the Fourier coefficients that  $\sum_{n=0}^\infty |\widehat{\phi\mu}(n)|^2 < \infty$  for all  $\phi \in L^2(\mu)$ , then  $\mu = \psi dm$  where  $m$  is Lebesgue measure on the unit circle and  $\psi \in L^\infty(m)$  (thanks to the referee for that observation) which we now show prevents  $N_\mu$  from being 2-weakly supercyclic.

**Theorem 8.1.** If  $T = M_z$  on  $L^2(\mu)$  where  $\mu = \psi dm$ , where  $m$  is Lebesgue measure on the unit circle,  $\psi \geq 0$ , and  $\psi \in L^\infty(m)$ , then  $T$  is not 2-weakly supercyclic.

**Proof.** Suppose that  $T$  is 2-weakly supercyclic and that  $\phi \in L^2(\mu)$  is a 2-weakly supercyclic vector for  $T$ . We may assume that  $\|\phi\|_{L^2(\mu)} = 1$ . Let  $f = 1$  and let us apply the weak angle criterion (Theorem 7.10). Notice that

$$\sum_{n=0}^{\infty} \left( \frac{|\langle T^n \phi, f \rangle|}{\|T^n \phi\|} \right)^2 = \sum_{n=0}^{\infty} |\langle T^n \phi, 1 \rangle|^2 = \sum_{n=0}^{\infty} \left| \int z^n \phi \cdot \psi dm \right|^2 = \sum_{n=0}^{\infty} |\widehat{(\phi\psi)}(n)|^2 < \infty$$

where in the first equality we used the fact that  $T$  is an isometry and thus  $\|T^n\phi\| = \|\phi\| = 1$  and in the last inequality we use the fact that  $\psi \in L^\infty(m)$  and thus  $\phi\psi \in L^2(m)$  (thus its Fourier coefficients are square summable). It now follows from Theorem 7.10 that  $\phi$  is not a 2-weakly supercyclic vector for  $T$ , a contradiction. Thus,  $T$  is not 2-weakly supercyclic.  $\square$

An operator  $T$  on a Hilbert space  $\mathcal{H}$  is said to be hyponormal if  $\|T^*x\| \leq \|Tx\|$  for all  $x \in \mathcal{H}$ . An operator is cohyponormal if its adjoint is hyponormal. In [16] it was shown that if  $T$  is hyponormal, then  $T^*$  is hypercyclic if and only if every part of the spectrum of  $T$  intersects both sides of the unit circle. That is,  $T^*$  is hypercyclic if and only if for every hyperinvariant subspace  $\mathcal{M}$  of  $T$  we have  $\sigma(T|_{\mathcal{M}}) \cap \{z \in \mathbb{C} : |z| < 1\} \neq \emptyset$  and  $\sigma(T|_{\mathcal{M}}) \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$ .

In [16] it was also shown that if  $T$  is hyponormal, then  $T^*$  is supercyclic if and only if there exists an  $r \geq 0$  such that one of the following holds:

- (1) for every hyperinvariant subspace  $\mathcal{M}$  of  $T$  we have  $\sigma(T|_{\mathcal{M}}) \cap \{z \in \mathbb{C} : |z| = r\} \neq \emptyset$  and  $\sigma(T|_{\mathcal{M}}) \cap \{z \in \mathbb{C} : |z| < r\} \neq \emptyset$ ;  
or
- (2) for every hyperinvariant subspace  $\mathcal{M}$  of  $T$  we have  $\sigma(T|_{\mathcal{M}}) \cap \{z \in \mathbb{C} : |z| = r\} \neq \emptyset$  and  $\sigma(T|_{\mathcal{M}}) \cap \{z \in \mathbb{C} : |z| < r\} \neq \emptyset$ .

If (1) holds we say that  $T$  is an **inner** supercyclic operator with supercyclicity radius  $r$  and if (2) holds we say that  $T$  is **outer** supercyclic operator with supercyclicity radius  $r$ . The circle itself is called a **supercyclicity circle** for  $T$  and its radius is a **supercyclicity radius** for  $T$ . Notice that the supercyclicity circle and the supercyclicity radius of a supercyclic operator are not necessarily unique. For instance, if  $B$  is the backward unilateral shift, then every circle centered at the origin with radius  $0 \leq r \leq 1$  is a supercyclicity circle for  $B$ . Notice that  $B$  is inner with respect to  $\{z : |z| = 1\}$  and  $B$  is outer with respect to  $\{z : |z| = 0\}$  and  $B$  is both inner and outer with respect to  $\{z : |z| = r\}$  when  $0 < r < 1$ .

However, if  $r \geq 0$  and  $T_b = \bigoplus_{n=1}^{\infty} (rI + \frac{1}{n}B)$ ,  $T_i = \bigoplus_{n=1}^{\infty} ((r - \frac{1}{n})I + \frac{1}{n}B)$ , and  $T_o = \bigoplus_{n=1}^{\infty} ((r + \frac{1}{n})I + \frac{1}{n}B)$ , then  $T_i$ ,  $T_o$ , and  $T_b$  are all supercyclic, they all have unique supercyclicity circles (with radius  $r$ ) and  $T_i$  is inner (if  $r > 0$ ),  $T_o$  is outer, and  $T_b$  is both inner and outer with respect to their supercyclicity circles.

**Proposition 8.2.** *If  $T$  is a hyponormal operator such that  $T^*$  is 1-weakly hypercyclic, then  $T^*$  is an outer supercyclic operator with supercyclicity radius 1.*

**Proof.** If  $T$  is hyponormal and  $T^*$  is 1-weakly hypercyclic, then by Corollary 3.4  $T$  must be a pure hyponormal operator. Also, if  $\mathcal{M}$  is an invariant subspace for  $T$ , then since  $T^*$  is 1-weakly hypercyclic we know from Proposition 3.2 that  $(T|_{\mathcal{M}})^*$  is also 1-weakly hypercyclic. Thus by Proposition 3.3  $\sigma((T|_{\mathcal{M}})^*)$  must intersect the unit circle, but this implies that  $\sigma(T|_{\mathcal{M}})$  also intersects the unit circle. Also, it must be the case that  $\sigma(T|_{\mathcal{M}}) \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$ , otherwise  $\sigma(T|_{\mathcal{M}}) \subseteq \text{cl } \mathbb{D}$ , which implies that all the orbits of  $(T|_{\mathcal{M}})^*$  are bounded, contradicting the fact that  $(T|_{\mathcal{M}})^*$  is 1-weakly hypercyclic. It now follows from Feldman, Miller and Miller [16, Theorem 7.5] that  $T^*$  is outer supercyclic with the unit circle being its supercyclicity circle.  $\square$

## 9. Universal families

In this section we extend the idea of an  $n$ -weakly hypercyclic operator to sequences of operators instead of just powers of a single operator. A commuting sequence  $\{T_n\}_{n=1}^{\infty}$  of operators

on a space  $X$  is said to be *universal* or a *hypercyclic sequence* if there is an  $x \in X$  such that  $\text{Orb}(x, \{T_n\}_{n=1}^\infty) = \{T_n x : n \geq 1\}$  is dense in  $X$ . If  $\text{Orb}(x, \{T_k\}_{k=1}^\infty)$  is  $n$ -weakly dense in  $X$ , then we will say that  $\{T_k\}_{k=1}^\infty$  is  *$n$ -weakly universal* or an  *$n$ -weakly hypercyclic sequence*.

**Definition 9.1.** A commuting sequence of operators  $\{T_n\}_{n=1}^\infty$  on a separable Banach space  $X$  is said to satisfy the *universality criterion* if there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, two dense sets  $D_1, D_2 \subseteq X$ , and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1)  $T_{n_k} x \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x \in D_1$ .
- (2)  $S_{n_k} y \rightarrow 0$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .
- (3)  $T_{n_k} S_{n_k} y \rightarrow y$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .

More accurately, we say that  $T$  satisfies the universality criterion with respect to the sequence  $\{n_k\}$  if the above conditions hold.

It is easy to show (see Theorem 9.2 below) that if  $\{T_n\}_{n=1}^\infty$  satisfies the universality criterion, then there exists a dense  $G_\delta$  set  $\Omega \subseteq X$  such that for each  $x \in \Omega$ ,  $\text{cl}[\text{Orb}(x, \{T_n\}_{n=1}^\infty)] = X$ . Bermúdez, Bonilla, and Peris [6] have shown that a commuting sequence  $T = \{T_k\}_{k=1}^\infty$  satisfies the universality criterion if and only if  $T \oplus T := \{T_k \oplus T_k\}_{k=1}^\infty$  is a universal sequence.

The following result is a simple generalization of the universality criterion given above where the set  $D_2$  is not required to be dense; it is a “universality” version of Theorem 6.3, its proof is the same as the proof of Theorem 6.3 except one replaces  $T^{n_k}$  with  $T_{n_k}$ .

**Theorem 9.2.** If  $\{T_n\}_{n=1}^\infty$  is a commuting sequence of operators on a separable Banach space  $X$  and there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, a dense set  $D_1 \subseteq X$ , and another (not necessarily dense) set  $D_2 \subseteq X$  and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1)  $T_{n_k} x \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x \in D_1$ ;
- (2) for each  $y \in D_2$ , there exists a subsequence  $\{n_{k_j}\}_{j=1}^\infty$  of  $\{n_k\}_{k=1}^\infty$  such that  $S_{n_{k_j}} y \rightarrow 0$  and  $T_{n_{k_j}} S_{n_{k_j}} y \rightarrow y$  as  $j \rightarrow \infty$ ,

then there exists a dense  $G_\delta$  set  $\Omega \subseteq X$  such that for each  $x \in \Omega$ , we have  $D_2 \subseteq \text{cl}[\text{Orb}(x, \{T_n\}_{n=1}^\infty)]$ .

**Proof.** We will apply Lemma 6.1. If  $U$  and  $V$  are any two open sets with  $V \cap D_2 \neq \emptyset$ , then let  $y \in V \cap D_2$  and since  $D_1$  is dense, we have  $U \cap D_1 \neq \emptyset$ , so choose an  $x \in U \cap D_1$ . By property (2), there is a subsequence  $\{n_{k_j}\}$  with the stated properties. Then for large  $j$ ,  $(x + S_{n_{k_j}} y)$  belongs to  $U$  and  $T_{n_{k_j}}(x + S_{n_{k_j}} y) = T_{n_{k_j}} x + T_{n_{k_j}} S_{n_{k_j}} y$  will belong to  $V$ . The theorem now follows by Lemma 6.1.  $\square$

**Definition 9.3.** A commuting sequence of operators  $\{T_n\}_{n=1}^\infty$  on a separable Banach space  $X$  is said to satisfy the *strong universality criterion* if there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, two dense sets  $D_1, D_2 \subseteq X$ , and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1)  $T_n x \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in D_1$ .
- (2)  $S_{n_k} y \rightarrow 0$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .
- (3)  $T_{n_k} S_{n_k} y \rightarrow y$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .

More accurately, we say that  $T$  satisfies the strong universality criterion with respect to the sequence  $\{n_k\}$  if the above conditions hold.

In [6] Bermúdez, Bonilla, and Peris prove the following result which we need, but first a definition is needed. If  $\{T_n\}_{n=1}^\infty$  is a commuting sequence of operators on  $X$  and  $y \in X$ , then a *backward orbit* for  $y$  (if it exists) is a sequence  $\{x_n\}_{n=1}^\infty \subseteq X$  such that  $T_n x_n = y$  for all  $n \geq 1$ . Recall that a precompact set is one whose closure is compact.

**Theorem 9.4.** *If  $\{T_n\}_{n=1}^\infty$  is a universal sequence of commuting operators on a separable Banach space  $X$  such that either*

- (1) *there is a dense set of vectors whose orbit is precompact, or*
- (2) *there is a dense set of vectors that admit precompact backward orbits,*

*then  $\{T_n\}_{n=1}^\infty$  satisfies the universality criterion.*

We will be interested in direct sums of universal sequences. If  $T_1 = \{T_{1,n}\}_{n=1}^\infty$  and  $T_2 = \{T_{2,n}\}_{n=1}^\infty$  are two sequences of commuting operators, then define the direct sum of  $T_1$  and  $T_2$  to be the commuting sequence  $T_1 \oplus T_2 = \{T_{1,n} \oplus T_{2,n}\}_{n=1}^\infty$ . A similar definition applies to finite or infinite direct sums of sequences of operators.

**Lemma 9.5.** *If  $T_1 = \{T_{1,n}\}_{n=1}^\infty, T_2 = \{T_{2,n}\}_{n=1}^\infty, \dots, T_p = \{T_{p,n}\}_{n=1}^\infty$  are each commuting sequences of operators satisfying the strong universality criterion and if their direct sum  $T = T_1 \oplus T_2 \oplus \dots \oplus T_p$  is a universal sequence, then  $T$  also satisfies the strong universality criterion.*

**Proof.** Since each sequence  $T_k = \{T_{k,n}\}_{n=1}^\infty$  satisfies the strong universality criterion, it has a dense set of vectors whose (full) orbits converge to zero, hence the same is true for their direct sum  $T$ . Since  $T$  is assumed to be universal, then Proposition 9.4 tells us that  $T$  satisfies the universality criterion. And since  $T$  has a dense set of vectors whose orbits converge to zero, it follows that  $T$  actually satisfies the strong universality criterion.  $\square$

**Theorem 9.6.** *Suppose that for each  $1 \leq j \leq p$ ,  $T_j = \{T_{j,k}\}_{k=1}^\infty$  is a commuting sequence of bounded linear operators on a separable Banach space  $X_j$  that satisfies the strong universality criterion. Suppose that  $1 \leq n < p$  and that the direct sum of any  $n$  of the sequences  $\{T_j\}_{j=1}^p$  is universal, then the direct sum  $T = \bigoplus_{j=1}^p T_j$  is  $n$ -weakly universal.*

The proof of this theorem is almost identical to that of Theorem 6.10, simply replace  $T_j^n$  by  $T_{j,n}$  and use Lemma 9.5 instead of Lemma 6.7. We leave the details to the reader.

## 10. Criteria for $n$ -weakly supercyclic operators

In this section we will give some criteria for the direct sum of a collection of supercyclic operators to be  $n$ -weakly supercyclic, similar to that done in Section 6 for  $n$ -weakly hypercyclic



operators. In particular, we will give necessary and sufficient conditions for the direct sum of a certain class of bilateral weighted shifts to be  $n$ -weakly supercyclic and this will allow us to produce easy examples of operators that are  $n$ -weakly supercyclic but not  $(2n)$ -weakly supercyclic.

There are a few different criteria for an operator to be supercyclic. In [6] it was shown that several of these criteria are all equivalent to the supercyclicity criterion that is stated below. It was also shown in [6] that  $T$  satisfies the supercyclicity criterion (below) if and only if  $T \oplus T$  is supercyclic.

**Definition 10.1.** An operator  $T$  on a separable  $F$ -space  $X$  is said to satisfy the *supercyclicity criterion* if there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, a sequence of non-zero scalars  $\{c_{n_k}\}_{n=1}^\infty$ , two dense sets  $D_1, D_2 \subseteq X$ , and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1)  $c_{n_k} T^{n_k} x \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x \in D_1$ .
- (2)  $(1/c_{n_k}) S_{n_k} y \rightarrow 0$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .
- (3)  $T^{n_k} S_{n_k} y \rightarrow y$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .

More accurately, we say that  $T$  satisfies the supercyclicity criterion with respect to the sequences  $\{n_k\}_{k=1}^\infty$  and  $\{c_{n_k}\}_{k=1}^\infty$  if the above conditions hold.

Notice that  $T$  satisfies the supercyclicity criterion if and only if there is an increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, and a sequence of non-zero scalars  $\{c_{n_k}\}_{k=1}^\infty$  such that the sequence  $\{c_{n_k} T^{n_k}\}_{k=1}^\infty$  satisfies the universality criterion.

**Definition 10.2.** An operator  $T$  on a separable Banach space  $X$  is said to satisfy the *strong supercyclicity criterion* if there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, a sequence of non-zero scalars  $\{c_n\}_{n=1}^\infty$ , two dense sets  $D_1, D_2 \subseteq X$ , and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1)  $c_n T^n x \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in D_1$ .
- (2)  $(1/c_{n_k}) S_{n_k} y \rightarrow 0$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .
- (3)  $T^{n_k} S_{n_k} y \rightarrow y$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .

Notice that  $T$  satisfies the strong supercyclicity criterion if and only if there exists a sequence  $\{c_n\}_{n=1}^\infty$  of non-zero scalars such that  $\{c_n T^n\}_{n=1}^\infty$  satisfies the strong universality criterion.

Thus we get the following result immediately from Theorem 9.6.

**Theorem 10.3** (Direct sums are  $n$ -weakly supercyclic). Suppose that for each  $1 \leq j \leq p$ ,  $T_j$  is a continuous linear operator on a separable Banach space  $X_j$ . Suppose also that there exists a sequence of non-zero scalars  $\{c_k\}_{k=1}^\infty$  such that each  $T_j$  satisfies the strong supercyclicity criterion with respect to the sequence  $\{c_k\}_{k=1}^\infty$ . Suppose also that  $1 \leq n < p$  and that the direct sum of any  $n$  of the operators  $\{T_j\}_{j=1}^p$  satisfies the supercyclicity criterion with respect to the sequence  $\{c_k\}_{k=1}^\infty$ , then  $T = \bigoplus_{j=1}^p T_j$  is  $n$ -weakly supercyclic.

**Proof.** Simply apply Theorem 9.6 to the sequences  $T_j = \{c_k T_j^k\}_{k=1}^\infty$ ,  $1 \leq j \leq p$ . The hypothesis that the direct sum of any  $n$  of the operators satisfies the supercyclicity criterion with respect

to the sequence  $\{c_k\}_{k=1}^\infty$  implies that the direct sum of any  $n$  of the sequences  $T_j$  satisfies the universality criterion and hence is universal. Thus Theorem 9.6 applies.  $\square$

The following criterion was given by Salas [23] for an operator to be supercyclic and was the first “supercyclicity criterion”. Salas’ criterion (below) is equivalent to the criterion in Definition 10.1. Both of these criteria for an operator  $T$  to be supercyclic are actually equivalent to  $T \oplus T$  being supercyclic (see [6]). Salas’ criterion is often easier to apply.

**Definition 10.4** (*Salas’ supercyclicity criterion*). An operator  $T$  on a separable Banach space  $X$  satisfies the *supercyclicity criterion* if there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, two dense sets  $D_1, D_2 \subseteq X$  and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1)  $\|T^{n_k}x\| \|S_{n_k}y\| \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x \in D_1$  and for all  $y \in D_2$ .
- (2)  $T^{n_k}S_{n_k}y \rightarrow y$  as  $k \rightarrow \infty$  for all  $y \in D_2$ .

We say that  $T$  satisfies the (Salas) supercyclicity criterion with respect to the sequence  $\{n_k\}_{k=1}^\infty$  if  $T$  satisfies the above conditions. In which case, there exists a dense  $G_\delta$  set  $\Omega \subseteq X$  such that  $\text{cl}[\mathbb{F} \cdot \text{Orb}(x, T)] = X$  for each  $x \in \Omega$ . This is actually a special case of the following more general criterion where  $D_2$  is not required to be dense.

**Theorem 10.5** (*Supercyclic containment criterion*). If  $T$  is an operator on a separable Banach space  $X$  and there exist a strictly increasing sequence  $\{n_k\}_{k=1}^\infty$  of positive integers, a dense set  $D_1$  in  $X$ , and a (not necessarily dense) set  $D_2 \subseteq X$  and functions  $S_{n_k} : D_2 \rightarrow X$  satisfying the following conditions:

- (1) for each  $y \in D_2$ , there exists a subsequence  $\{n_{k_j}\}_{j=1}^\infty$  of  $\{n_k\}_{k=1}^\infty$  such that
  - (a)  $\|T^{n_{k_j}}x\| \|S_{n_{k_j}}y\| \rightarrow 0$  as  $j \rightarrow \infty$  for all  $x \in D_1$  and
  - (b)  $T^{n_{k_j}}S_{n_{k_j}}y \rightarrow y$  as  $j \rightarrow \infty$ ,

then there exists a dense  $G_\delta$  set  $\Omega \subseteq X$  such that  $D_2 \subseteq \text{cl}[\mathbb{F} \cdot \text{Orb}(x, T)]$  for each  $x \in \Omega$ .

**Proof.** We will apply Lemma 6.1. If  $U$  and  $V$  are any two open sets with  $V \cap D_2 \neq \emptyset$ , then let  $y \in V \cap D_2$ . Since  $D_1$  is dense we may choose an  $x \in U \cap D_1$ . Since property (a) holds, an elementary argument (see [5, p. 9]) shows that we can find scalars  $\{c_j\}$  such that  $c_j T^{n_{k_j}}x \rightarrow 0$  and  $\frac{1}{c_j} S_{n_{k_j}}y \rightarrow 0$  as  $j \rightarrow \infty$ . Then for large  $j$ ,  $(x + \frac{1}{c_j} S_{n_{k_j}}y)$  belongs to  $U$  and  $c_j T^{n_{k_j}}(x + \frac{1}{c_j} S_{n_{k_j}}y) = c_j T^{n_{k_j}}x + T^{n_{k_j}}S_{n_{k_j}}y$  will belong to  $V$ . The theorem now follows by a version of Lemma 6.1.  $\square$

For the next example you may want to see Section 11 for some basic background on hypercyclicity and supercyclicity of weighted shifts.

**Example 10.6** (*Direct sums of bilateral weighted shifts*). Suppose that  $\{T_i\}_{i=1}^p$  are backward bilateral weighted shifts on  $\ell^2(\mathbb{Z})$  with weight sequences  $\{w_{i,k}\}_{k=-\infty}^\infty$ ,  $1 \leq i \leq p$ . Let  $1 \leq n < p$  and suppose that the following hold:

- (1) The products of the negative weights are bounded above and below. That is, there exist an  $\varepsilon, M > 0$  such that  $0 < \varepsilon \leq \prod_{k=-m}^{-1} w_{i,k} \leq M < \infty$  for all  $m > 0$  and  $1 \leq i \leq p$ .
- (2) The direct sum of any  $n$  of the operators  $\{T_i\}_{i=1}^p$  is supercyclic.

Then  $T = \bigoplus_{i=1}^p T_i$  is  $n$ -weakly supercyclic.

Notice that condition (1) holds if, for example,  $w_{i,k} = 1$  for all  $k \leq 0$  and all  $1 \leq i \leq p$ .

**Proof of Example 10.6.** We will verify the conditions of Theorem 10.5. Let  $\mathcal{F}$  be the set of all vectors in  $\ell^2(\mathbb{N})$  with finite support, that is, vectors with at most finitely many non-zero coordinates and let  $\mathcal{H} = \ell^2(\mathbb{Z})^p = \bigoplus_{i=1}^p \ell^2(\mathbb{Z})$ . Now define

$$D_1 = \mathcal{F}^p = \bigoplus_{i=1}^p \mathcal{F} = \{(x_i)_{i=1}^p \in \mathcal{H} : x_i \in \mathcal{F} \text{ for } 1 \leq i \leq p\}.$$

Notice by assumption (1) in our hypothesis, we have that

$$(i) \quad 0 < \inf_{n \geq 0} \|T^n x\| \leq \sup_{n \geq 0} \|T^n x\| < \infty \quad \text{for all } x \in D_1 \setminus \{0\}.$$

For any  $J \subseteq \{1, \dots, p\}$  with  $|J| \leq n$ , define

$$\mathcal{H}_J = \{(x_i)_{i=1}^p \in \mathcal{H} : x_i = 0 \text{ if } i \notin J\}.$$

Also define  $\hat{\mathcal{H}}_J = \bigoplus_{i \in J} \ell^2(\mathbb{Z})$ . Clearly  $\mathcal{H}_J$  and  $\hat{\mathcal{H}}_J$  are naturally isomorphic. If  $T_J = \bigoplus_{i \in J} T_i$ , then we may naturally consider  $T_J$  as an operator on  $\mathcal{H}_J$  or  $\hat{\mathcal{H}}_J$ .

Let  $S_i$  be the right inverse of  $T_i$  (so  $S_i$  is a forward weighted shift with weight sequence  $\{1/w_{i,k}\}_{k=1}^\infty$ , although  $S_i$  may not be bounded) and let  $S = \bigoplus_{i=1}^p S_i$ , then  $TS = I$ . Let  $S_k = S^k$  for all  $k \geq 1$ .

By assumption, if  $|J| \leq n$ , then  $T_J$  is supercyclic on  $\mathcal{H}_J$  and so by Salas' Theorem (Theorem 11.2)  $T_J$  satisfies the (Salas) supercyclicity criterion (see Definition 10.4). Thus there exist dense sets  $D_{1,J}, D_{2,J}$  in  $\mathcal{H}$  and a strictly increasing sequence of integers  $\{n_{k,J}\}_{k=1}^\infty$  satisfying the conditions of Salas' supercyclicity criterion. In fact Theorem 11.2 implies that we can choose  $D_{1,J} = D_{2,J} = \mathcal{F}_J := \mathcal{F}^p \cap \mathcal{H}_J$ . Thus for the sequence  $\{n_{k,J}\}_{k=1}^\infty$  and for any  $x, y \in \mathcal{F}_J$  we have

$$(ii) \quad \|T^{n_{k,J}} x\| \|S_{n_{k,J}} y\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

By (i) above we know that  $\{\|T^{n_{k,J}} x\|\}$  is bounded away from zero and thus we must have that for any  $y \in \mathcal{F}_J$ ,

$$(iii) \quad \|S_{n_{k,J}} y\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Now we have already defined  $D_1 = \mathcal{F}^p$ , a dense set in  $\mathcal{H}$ , and let us define  $D_2$  as follows:

$$D_2 = \{(x_i)_{i=1}^p \in \mathcal{F}^p : |\{i : x_i \neq 0\}| \leq n\} = \bigcup \{\mathcal{F}_J : J \subseteq \{1, \dots, p\} \text{ \& } |J| \leq n\}.$$

By passing to subsequences if necessary, we may assume that the sequences  $\{n_{k,J}\}_{k=1}^\infty$  are disjoint for different sets  $J$ . Then let  $\{n_k\}_{k=1}^\infty$  be an increasing enumeration of  $\bigcup\{n_{k,J}: J \subseteq \{1, \dots, p\}, |J| \leq n\}$ .

We may now check condition (1)(a) of Theorem 10.5. Let  $y \in D_2$ , then there exists a  $J \subseteq \{1, \dots, p\}$  with  $|J| \leq n$  such that  $y \in \mathcal{F}_J$ . Then  $\{n_{k,J}\}_{k=1}^\infty$  is a subsequence of  $\{n_k\}_{k=1}^\infty$  and by (iii) we know that  $S_{n_{k,J}}y \rightarrow 0$  as  $k \rightarrow \infty$ . This together with the fact from (i) that  $\{\|T^{n_{k,J}}x\|\}$  is bounded for any  $x \in D_1$ , gives

$$(iv) \quad \|T^{n_{k,J}}x\| \|S_{n_{k,J}}y\| \rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ for all } x \in D_1 \text{ and } y \in D_2.$$

Thus condition (1)(a) holds. Notice that the difference between (ii) and (iv) is that in (ii) we must have  $x \in \mathcal{F}_J = D_{1,J}$  where as in (iv)  $x$  may be in the larger set  $D_1 = \mathcal{F}$ .

Also, since  $S$  is a right inverse of  $T$  condition (1)(b) of Theorem 10.5 holds trivially.

Since (1)(a) and (1)(b) of Theorem 10.5 both hold, it follows that there is a dense  $G_\delta$  set  $\Omega \subseteq \mathcal{H}$  such that for any  $x \in \Omega$  we have  $D_2 \subseteq \text{cl}[\mathbb{F} \cdot \text{Orb}(x, T)]$ . Now by Proposition 6.8 the set  $D_2$  is  $n$ -weakly dense in  $\mathcal{H}$ . It follows that  $T$  is  $n$ -weakly supercyclic, as desired.  $\square$

The previous theorem may be combined with the following theorem to show that certain direct sums of bilateral weighted shifts are  $n$ -weakly supercyclic but not weakly supercyclic. See the corollary afterward.

**Theorem 10.7** (*Direct sums of bilateral weighted shifts*). Suppose that  $\{T_i\}_{i=1}^n$  are backward bilateral weighted shifts on  $\ell^2(\mathbb{Z})$ . If  $T = \bigoplus_{i=1}^n T_i$  is  $(2n)$ -weakly supercyclic, then  $T$  must be supercyclic.

The proof of this theorem is essentially due to Shkarin [24] (also see [5, p. 242]), except that we need to use the “ $n$ -weak angle criterion” (Theorem 7.8) and the following simple lemma.

**Lemma 10.8.** If  $\{a_1, a_2, \dots, a_p\}$  and  $\{b_1, b_2, \dots, b_p\}$  are finite sets of non-negative real numbers, then

$$\min\{a_1b_1, a_2b_2, \dots, a_pb_p\} \leq \min\{a_1, a_2, \dots, a_p\} \cdot \max\{b_1, b_2, \dots, b_p\},$$

and

$$\max\{a_1b_1, a_2b_2, \dots, a_pb_p\} \geq \max\{a_1, a_2, \dots, a_p\} \cdot \min\{b_1, b_2, \dots, b_p\}.$$

**Proof.** Let  $M = \max\{b_1, b_2, \dots, b_p\}$ , then since the  $a_k$ 's are non-negative we have that for every  $1 \leq k \leq p$

$$a_kb_k \leq a_kM.$$

Then since  $M \geq 0$ ,

$$\begin{aligned} \min\{a_1b_1, a_2b_2, \dots, a_pb_p\} &\leq \min\{a_1M, a_2M, \dots, a_pM\} = M \min\{a_1, a_2, \dots, a_p\} \\ &= \min\{a_1, a_2, \dots, a_p\} \cdot \max\{b_1, b_2, \dots, b_p\}. \end{aligned}$$

Similarly, if we let  $m = \min\{b_1, b_2, \dots, b_p\}$ , then since the  $a_k$ 's are non-negative we have that for every  $1 \leq k \leq p$

$$a_k b_k \geq a_k m.$$

Then since  $m \geq 0$ ,

$$\begin{aligned} \max\{a_1 b_1, a_2 b_2, \dots, a_p b_p\} &\geq \max\{a_1 m, a_2 m, \dots, a_p m\} = m \max\{a_1, a_2, \dots, a_p\} \\ &= \max\{a_1, a_2, \dots, a_p\} \cdot \min\{b_1, b_2, \dots, b_p\}. \quad \square \end{aligned}$$

**Proof of Theorem 10.7.** Suppose that  $\{T_i\}_{i=1}^n$  are backward bilateral weighted shifts on  $\ell^2(\mathbb{Z})$  with weight sequences  $\{w_{i,k}\}_{k=-\infty}^\infty$ ,  $1 \leq i \leq n$ . Let  $T = \bigoplus_{i=1}^n T_i$  which acts on  $\mathcal{H} = \bigoplus_{j=1}^n \ell^2(\mathbb{Z})$ .

Suppose that  $T$  is  $(2n)$ -weakly supercyclic and by way of contradiction, suppose that  $T$  is not supercyclic. We will apply the “ $n$ -weak angle criterion” (Theorem 7.8).

Since  $T$  is not supercyclic, then by Theorem 11.2 we may find a  $q \geq 1$  and a  $\delta > 0$  such that

$$\max_{1 \leq i, j \leq n} \left\{ \frac{w_{i,q-k+1} \cdots w_{i,q}}{w_{j,q+1} \cdots w_{j,q+k}} \right\} \geq \delta \quad \text{for every } k \geq 1.$$

Thus by taking reciprocals we get

$$\min_{1 \leq i, j \leq n} \left\{ \frac{w_{j,q+1} \cdots w_{j,q+k}}{w_{i,q-k+1} \cdots w_{i,q}} \right\} = \frac{1}{\max_{1 \leq i, j \leq n} \left\{ \frac{w_{i,q-k+1} \cdots w_{i,q}}{w_{j,q+1} \cdots w_{j,q+k}} \right\}} \leq \frac{1}{\delta} \quad \text{for every } k \geq 1. \quad (4)$$

Let  $e_q = (\dots, 0, 0, 0, 1, 0, 0, \dots) \in \ell^2(\mathbb{Z})$  where the 1 is in the  $q$ th position and for  $1 \leq j \leq n$ , let  $f_j = (0, 0, \dots, e_q, 0, \dots, 0) \in \mathcal{H} = \bigoplus_{j=1}^n \ell^2(\mathbb{Z})$  where  $e_q$  is in the  $j$ th position of  $f_j$ . Since  $T$  is  $(2n)$ -weakly supercyclic, the set of  $(2n)$ -weakly supercyclic vectors forms a  $(2n)$ -weakly dense set in  $\mathcal{H}$ , and hence we can choose a  $(2n)$ -weakly supercyclic vector  $x = (x_1, \dots, x_n) \in \mathcal{H}$  for  $T$  such that

$$x_{j,q} := \langle x_j, e_q \rangle = \langle x, f_j \rangle \neq 0 \quad \text{for all } 1 \leq j \leq n. \quad (5)$$

This follows since every vector in the scaled orbit of a  $(2n)$ -weakly supercyclic vector is also a  $(2n)$ -weakly supercyclic vector and that the set  $V = \{x \in \mathcal{H} : \langle x, f_j \rangle \neq 0 \text{ for all } 1 \leq j \leq n\}$  is an  $n$ -weakly open set in  $\mathcal{H}$ .

Now since

$$\|T_j^k x_j\| \geq |\langle T_j^k x_j, e_{q-k} \rangle| = w_{j,q} w_{j,q-1} \cdots w_{j,q-k+1} |x_{j,q}| > 0$$

for all  $k \geq 1$  and all  $1 \leq j \leq n$ , we get from Lemma 10.8

$$\begin{aligned} \|T^k x\| &\geq \max_{1 \leq j \leq n} \|T_j^k x_j\| \geq \max_{1 \leq j \leq n} (w_{j,q} w_{j,q-1} \cdots w_{j,q-k+1} |x_{j,q}|) \\ &\geq \max_{1 \leq j \leq n} \{w_{j,q} w_{j,q-1} \cdots w_{j,q-k+1}\} \min_{1 \leq j \leq n} \{|x_{j,q}|\} > 0. \end{aligned}$$

Notice that

$$f_j(T^k x) := \langle T^k x, f_j \rangle = \langle T_j^k x_j, e_q \rangle = w_{j,q+1} w_{j,q+2} \cdots w_{j,q+k} \cdot x_{j,q+k}.$$

Thus using this and Lemma 10.8 we have the following:

$$\begin{aligned} & \sum_{k=0}^{\infty} \left( \frac{\min\{|f_1(T^k x)|, |f_2(T^k x)|, \dots, |f_n(T^k x)|\}}{\|T^k x\|} \right)^2 \\ &= \sum_{k=0}^{\infty} \left( \frac{\min_{1 \leq j \leq n} \{w_{j,q+1} w_{j,q+2} \cdots w_{j,q+k} \cdot |x_{j,q+k}|\}}{\|T^k x\|} \right)^2 \\ &\leq \sum_{k=0}^{\infty} \left( \frac{\min_{1 \leq j \leq n} \{w_{j,q+1} w_{j,q+2} \cdots w_{j,q+k}\} \max_{1 \leq j \leq n} \{|x_{j,q+k}|\}}{\max_{1 \leq j \leq n} \{w_{j,q} w_{j,q-1} \cdots w_{j,q-k+1}\} \min_{1 \leq j \leq n} \{|x_{j,q}|\}} \right)^2 \\ &= \frac{1}{\min_{1 \leq j \leq n} \{|x_{j,q}|^2\}} \sum_{k=0}^{\infty} \left( \min_{1 \leq i, j \leq n} \left\{ \frac{w_{j,q+1} w_{j,q+2} \cdots w_{j,q+k}}{w_{i,q} w_{i,q-1} \cdots w_{i,q-k+1}} \right\} \cdot \max_{1 \leq j \leq n} |x_{j,q+k}| \right)^2 \\ &\leq \frac{1}{\delta^2 \min_{1 \leq j \leq n} \{|x_{j,q}|^2\}} \sum_{k=0}^{\infty} \max_{1 \leq j \leq n} |x_{j,q+k}|^2 \leq \frac{\|x\|^2}{\delta^2 \min_{1 \leq j \leq n} \{|x_{j,q}|^2\}} < \infty. \end{aligned}$$

Since the above sum is convergent, then according to the  $n$ -weak angle criterion (Theorem 7.8), the vector  $x$  is not a  $(2n)$ -weakly supercyclic vector for  $T$ . However, this contradicts our choice of  $x$ . Thus it must be true that  $T$  is supercyclic.  $\square$

**Corollary 10.9.** *For any positive integer  $n$ , there exist bilateral weighted shifts  $T_1, T_2, \dots, T_{n+1}$  such that  $T = \bigoplus_{k=1}^{n+1} T_k$  is  $n$ -weakly supercyclic, but not  $(2n+2)$ -weakly supercyclic.*

**Proof.** Define  $w_{i,k} = 1$  for all  $k \leq 0$  and for all  $1 \leq i \leq (n+1)$  (we will define the positive weights shortly). Then if  $T_i$  is the bilateral weighted shift on  $\ell^2(\mathbb{Z})$  with weight sequence  $\{w_{i,k}\}_{k=-\infty}^{\infty}$ , then  $T_i$  will be supercyclic precisely when the backward unilateral weighted shift  $\hat{T}_i$  with weight sequence  $\{w_{i,k}\}_{k=1}^{\infty}$  is hypercyclic on  $\ell^2(\mathbb{N})$  (see Theorem 11.1). Now simply choose the positive weights (as in Corollary 6.14) so that the direct sum of any  $n$  of the operators  $\{\hat{T}_1, \hat{T}_2, \dots, \hat{T}_{n+1}\}$  is hypercyclic, but the direct sum of all  $(n+1)$  of the operators is not hypercyclic.  $\square$

## 11. Hypercyclicity and supercyclicity of weighted shifts

This section is for the convenience and easy reference for the reader. If  $\{e_k\}_{k=-\infty}^{\infty}$  is the standard basis for  $\ell^2(\mathbb{Z})$ , then a backward bilateral weighted shift  $T$  on  $\ell^2(\mathbb{Z})$  acts as follows:  $Te_k = w_k e_{k-1}$  for all  $k \in \mathbb{Z}$ . The shift  $T$  has a unique right inverse (although it may not be bounded):  $Se_k = \frac{1}{w_{k+1}} e_{k+1}$  for all  $k \in \mathbb{Z}$ . Clearly  $TS = I$ . The following also hold:

$$\|T^m e_k\| = (w_{k-m+1} \cdots w_k) \quad \text{and} \quad \|S^m e_k\| = \frac{1}{w_{k+1} \cdots w_{k+m}}.$$

The next result is in Shkarin [24] or [5, p. 242] and is a simple variation of Salas' criterion characterizing hypercyclicity and supercyclicity of weighted shifts.

**Theorem 11.1.**

- (a) If  $T$  is a bilateral backward weighted shift on  $\ell^2(\mathbb{Z})$  with weight sequence  $\{w_n\}_{n \in \mathbb{Z}}$ , then  
 (1)  $T$  is hypercyclic if and only if

$$\text{for every } k \geq 1, \quad \liminf_{m \rightarrow \infty} \max \left\{ \frac{1}{w_{k+1} \cdots w_{k+m}}, (w_{k-m+1} \cdots w_k) \right\} = 0,$$

- (2)  $T$  is supercyclic if and only if

$$\text{for every } k \geq 1, \quad \liminf_{m \rightarrow \infty} \frac{w_{k-m+1} \cdots w_k}{w_{k+1} \cdots w_{k+m}} = 0.$$

- (b) If  $T$  is a unilateral backward weighted shift on  $\ell^2(\mathbb{N})$  with weight sequence  $\{w_n\}_{n=1}^\infty$ , then  $T$  is hypercyclic if and only if there exists a sequence  $\{n_k\}_{k=1}^\infty$  of positive integers such that  $(w_1 w_2 \cdots w_{n_k}) \rightarrow \infty$  as  $k \rightarrow \infty$ .

The following result is a combination of Salas' criterion for direct sums to be supercyclic and the above characterization of supercyclicity for shifts.

**Theorem 11.2 (Salas/Shkarin).** If  $T_j$  is a bilateral weighted shift on  $\ell^2(\mathbb{Z})$  with weight sequence  $\{w_{j,k}\}_{k=-\infty}^\infty$  for each  $1 \leq j \leq n$ , then  $T = \bigoplus_{j=1}^n T_j$  is supercyclic if and only if

$$\text{for every } k \geq 1, \quad \liminf_{m \rightarrow \infty} \max \left\{ \frac{w_{i,k-m+1} \cdots w_{i,k}}{w_{j,k+1} \cdots w_{j,k+m}} : 1 \leq i, j \leq n \right\} = 0.$$

The previous condition can be restated as

$$\text{for every } k \geq 1, \quad \liminf_{m \rightarrow \infty} \max \{ \|T_i^m e_k\| \|S_j^m e_k\| : 1 \leq i, j \leq n \} = 0.$$

## 12. Questions

- (1) For which symbols  $f \in H^\infty(\mathbb{D})$  is the coanalytic Toeplitz operator  $T_f^*$  1-weakly hypercyclic on  $H^2(\mathbb{D})$ ?
- (2) Conjecture: If  $G$  is a region bounded by a smooth Jordan curve  $\Gamma$  such that  $G$  is in the exterior of the open unit disk but  $\Gamma$  intersects the unit circle in a non-trivial arc, then  $M_z^*$  on  $H^2(G)$  is a 1-weakly hypercyclic operator.
- (3) Conjecture:  $T = 2I + B$  is not 1-weakly hypercyclic where  $B$  is the backward shift on  $\ell^2(\mathbb{N})$ .
- (4) Conjecture: For every  $n \geq 1$ , there are Hilbert space operators that are  $n$ -weakly supercyclic but not  $(n+1)$ -weakly supercyclic.
- (5) Is there an  $n$ -weakly somewhere dense theorem? In particular, if the  $n$ -weak closure of  $\text{Orb}(x, T)$  contains an  $n$ -weakly open set, then must  $\text{Orb}(x, T)$  be  $n$ -weakly dense?

- (6) If  $T$  is  $n$ -weakly hypercyclic, then is  $T^p$  also  $n$ -weakly hypercyclic for  $p \geq 1$ ? Note that by Theorem 5.8  $T$  and  $T^p$  need not have the same  $n$ -weak hypercyclic vectors.
- (7) (An  $n$ -weak version of Ansari's Theorem.) If  $\text{Orb}(x, T)$  is  $n$ -weakly dense in  $X$  and if  $(n+1)$  and  $p$  are relatively prime, then is  $\text{Orb}(x, T^p)$   $n$ -weakly dense in  $X$ ? See Theorem 5.8.
- (8) (A general  $n$ -weak version of Ansari's Theorem.) If  $\text{Orb}(x, T)$  is  $n$ -weakly dense in  $X$  and if  $d = \gcd(n+1, p)$ , then is  $\text{Orb}(x, T^p)$   $(\frac{n+1}{d} - 1)$ -weakly dense in  $X$ ? See Theorem 5.8.
- (9) Can a hyponormal operator be 2-weakly supercyclic?
- (10) If an operator has a dense set of vectors whose forward orbits converge to zero weakly and if every vector has a backward orbit that converges to zero in norm, then must the operator be 1-weakly hypercyclic (or actually weakly hypercyclic)?
- (11) Is there a 1-weakly hypercyclic operator that is not norm supercyclic?
- (12) For  $n \geq 1$ , is there an operator that is  $n$ -weakly hypercyclic, but not  $(n+1)$ -weakly supercyclic?
- (13) Is there an operator that is  $n$ -weakly hypercyclic for every  $n$ , but not weakly hypercyclic?
- (14) If  $I$  is the identity operator on a one-dimensional space, then when is  $T \oplus I$   $n$ -weakly supercyclic? In particular, are any of the implications in Theorem 7.6 necessary and sufficient?
- (15) If  $T_1$  and  $T_2$  both satisfy the strong hypercyclicity criterion (Definition 6.6) and  $T_1 \oplus T_2$  is 2-weakly hypercyclic, then must  $T_1 \oplus T_2$  be hypercyclic?
- (16) An operator  $T$  on a space  $X$  is  $N$ -supercyclic if there is an  $N$ -dimensional subspace whose orbit under  $T$  is dense in  $X$ . What are some examples of operators that are  $n$ -weakly  $N$ -supercyclic? That is, operators that have  $N$ -dimensional subspaces whose orbits are  $n$ -weakly dense.
- (17) Is a weakly sequentially hypercyclic operator necessarily hypercyclic?
- (18) If an orbit is weakly sequentially dense is it necessarily dense?
- (19) If an operator is weakly hypercyclic and there is a dense set of vectors whose orbits converge to zero in norm, then is the operator hypercyclic?
- (20) If  $T$  is weakly hypercyclic, then does the unit ball have dense orbit?

## Acknowledgments

The author would like to thank Paul Bourdon for his helpful conversations regarding this paper and the referee for his suggestions and corrections.

## References

- [1] S.I. Ansari, Hypercyclic and cyclic vectors, *J. Funct. Anal.* 128 (1995) 374–383.
- [2] K. Ball, The plank problem for symmetric bodies, *Invent. Math.* 104 (3) (1991) 535–543.
- [3] K. Ball, The complex plank problem, *Bull. Lond. Math. Soc.* 33 (4) (2001) 433–442.
- [4] F. Bayart, É. Matheron, Hyponormal operators, weighted shifts, and weak forms of supercyclicity, *Proc. Edinb. Math. Soc.* 49 (2006) 1–15.
- [5] F. Bayart, É. Matheron, *Dynamics of Linear Operators*, Cambridge University Press, 2009.
- [6] T. Bermúdez, A. Bonilla, A. Peris, On hypercyclicity and supercyclicity criteria, *Bull. Aust. Math. Soc.* 70 (2004) 45–54.
- [7] J. Bes, A. Peris, Hereditarily hypercyclic operators, *J. Funct. Anal.* 167 (1) (1999) 94–112.
- [8] P.S. Bourdon, Invariant manifolds of hypercyclic vectors, *Proc. Amer. Math. Soc.* 118 (3) (1993) 845–847.
- [9] P.S. Bourdon, N.S. Feldman, Somewhere dense orbits are everywhere dense, *Indiana Univ. Math. J.* 52 (3) (2003) 811–819.
- [10] P.S. Bourdon, N.S. Feldman, J.H. Shaprio, Some properties of  $n$ -supercyclic operators, *Studia Math.* 165 (2) (2004) 135–157.



- [11] J.B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1985.
- [12] J.B. Conway, *The Theory of Subnormal Operators*, Amer. Math. Soc., Providence, RI, 1991.
- [13] G. Costakis, On a conjecture of D. Herrero concerning hypercyclic operators, *C. R. Acad. Sci. Paris* 330 (2000) 179–182.
- [14] N.S. Feldman, Pure subnormal operators have a cyclic adjoint, *J. Funct. Anal.* 162 (2) (1999) 379–399.
- [15] N.S. Feldman,  $n$ -Weakly supercyclic matrices, preprint.
- [16] N.S. Feldman, T.L. Miller, V.G. Miller, Hypercyclic and supercyclic cohyponormal operators, *Acta Sci. Math. (Szeged)* 68 (3–4) (2002) 965–990.
- [17] K.G. Grosse-Erdmann, A. Peris Manguillot, *Linear Chaos*, Universitext, Springer-Verlag, 2011.
- [18] H.M. Hilden, L.J. Wallen, Some cyclic and non-cyclic vectors of certain operators, *Indiana Univ. Math. J.* 23 (1974) 557–565.
- [19] A. Montes-Rodriguez, S.A. Shkarin, Non-weakly supercyclic operators, *J. Operator Theory* 58 (1) (2007) 39–62.
- [20] A. Peris, Multihypercyclic operators are hypercyclic, *Math. Z.* 236 (2001) 779–786.
- [21] S. Rolewicz, On orbits of elements, *Studia Math.* 32 (1969) 17–22.
- [22] H.N. Salas, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc.* 347 (3) (1995) 993–1004.
- [23] H.N. Salas, Supercyclicity and weighted shifts, *Studia Math.* 135 (1) (1999) 55–74.
- [24] S. Shkarin, Non-sequential weak supercyclicity and hypercyclicity, *J. Funct. Anal.* 242 (2007) 37–77.
- [25] S. Shkarin, A weighted bilateral shift with cyclic square is supercyclic, *Bull. Lond. Math. Soc.* 39 (2007) 1029–1038.
- [26] J.E. Thomson, Approximation in the mean by polynomials, *Ann. of Math.* (2) 133 (3) (1991) 477–507.